ANISOTROPY OF ELASTIC PROPERTIES OF MATERIALS

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Papers dealing with the generalized Hooke’s law for linearly elastic anisotropic media are reviewed. The papers considered are based on Kelvin’s approach disclosing the structure of the generalized Hooke’s law, which is determined by six eigenmoduli of elasticity and six orthogonal eigenstates.

Key words: anisotropy, elasticity moduli, eigenmoduli and eigenstates, linearly elastic materials.

Many natural materials, such as rocks, crystals, and biological tissues, and also materials used in advanced technologies, in particular, composites, are characterized by substantial anisotropy of their elasticity properties. In most cases, composite materials, as well as their components, are anisotropic materials. To create composite materials with elasticity properties necessary for engineering practice, one should know admissible limits of the components of the tensor of the elasticity moduli and the tensor of the compliance coefficients of anisotropic materials.

The constitutive equations of the linear theory of elasticity \[1–5\] in the Cartesian rectangular coordinate system \((x_1, x_2, x_3)\) include the equations of motion

\[
\sigma_{ij,j} - \rho \frac{\partial^2 u_i}{\partial t^2} + F_i = 0, \quad i, j = 1, 2, 3, \tag{1}
\]

the generalized Hooke’s law

\[
\sigma_{ij} = E_{ijkl} \varepsilon_{kl}, \quad i, j, k, l = 1, 2, 3, \tag{2}
\]

and the Cauchy formulas, which express strains via displacements:

\[
2\varepsilon_{kl} = u_{l,k} + u_{k,l}. \tag{3}
\]

In Eqs. (1)–(3), \(\sigma_{ij} = \sigma_{ji}\) are the components of the symmetric stress tensor, \(\varepsilon_{ij} = \varepsilon_{ji}\) are the components of the strain tensor, \(E_{ijkl}\) are the components of the fourth-rank tensor of the elasticity moduli, \(u_i\) are the components of the displacement vector, \(F_i\) are the components of the vector of bulk forces, \(\rho\) is the constant density of the material, and \(t\) is the time. The comma ahead of the subscript indicates differentiation with respect to the spatial coordinate marked by this subscript; repeated letters in the subscripts indicate summation over their admissible values. Relations (2) can be inverted: \(\varepsilon_{ij} = S_{ijkl} \sigma_{kl}\) \((S_{ijkl}\) are the components of the fourth-rank tensor of the compliance coefficients).

In the linear theory of elasticity, the specific strain energy for anisotropic materials is presented as \[1, 2\]

\[
2\Phi = E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = S_{ijkl} \sigma_{ij} \sigma_{kl}. \tag{4}
\]

The components \(E_{ijkl}\) possess the properties of symmetry \[2, 5\]:

\[
E_{ijkl} = E_{jikl} = E_{klij}. \tag{5}
\]

The constants \(S_{ijkl}\) also satisfy the conditions of symmetry (5) and are related to \(E_{ijkl}\) by the expressions

\[
E_{ijkl} S_{klrs} = \delta_{ijrs} = (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr})/2, \quad S_{ijkl} E_{klrs} = \delta_{ijrs},
\]
where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The tensor $\delta_{ijrs}$ is a unit tensor in the space of symmetric tensors of the form (5).

The issues associated with the presentation of Hooke’s law (2) in special bases and identification of the limits of variation of the constants $E_{ijkl}$ compatible with the positive definiteness of the quadratic form (4) were considered in [1, 6–22]. The quadratic form (4) is reduced below to the canonical form, which allows understanding of the structure of the tensor $E_{ijkl}$.

Substituting relations (2) and (3) into Eq. (1), we obtain the equations of motion in displacements [2]

$$E_{ijkl} u_{j,kl} - \rho \frac{\partial^2 u_i}{\partial t^2} + F_i = 0,$$

where

$$E_{ijkl} = (E_{iklj} + E_{ilkj})/2.$$

The properties of the matrix $E_{ijkl}$ were considered in [23, 24]. In solving particular problems, Eqs. (1)–(3) or (6) are supplemented by initial and boundary conditions. Equations (1) and (6) yield the static equations in the form

$$\sigma_{ij} + F_i = 0, \quad E_{ijkl} u_{j,kl} + F_i = 0.$$

Let $n_i$ and $m_i$ ($i = 1, 2, 3$) be two orthogonal unit directions. Young’s modulus $E_n$ in the direction $n_i$ is determined in the form

$$1/E_n = n_i n_j S_{ijkl} n_k n_l.$$

Poisson’s ratio $\nu_{mn}$ in the direction $m_i$ under tension in the direction $n_i$ is

$$\nu_{mn}/E_n = -m_i m_j S_{ijkl} n_k n_l$$

(no summation in terms of $n$ is performed on the left). The shear modulus $\mu_{nm}$ between the areas with the normals $n_i$ and $m_i$ is

$$1/(4\mu_{nm}) = n_i n_j S_{ijkl} n_k m_l.$$

The bulk modulus $K$ can be presented in the form $1/K = S_{ik kk}$. The strain energy (4) should be a positively determined quadratic form [1, 3, 25].

The classical linear theory of elasticity was developed in the 19th century by A. L. Cauchy, C. L. M. H. Navier, S. D. Poisson, G. Green, A. J. C. Barré de Saint-Venant, and other scientists (the history of the elasticity theory is described in [3, 26–31]).

In 1660, R. Hooke discovered the law of proportionality of stresses and strains in the simplest form. Cherepanov [32] said that L. Euler was the first to formulate the law of elasticity in the form $\sigma = E \varepsilon$ [he denoted the constant $E$ by the first letter of his name (Euler)].

In 1821, C. L. M. H. Navier started constructing the elasticity theory. In 1822, A. L. Cauchy introduced the notion of a stress state (as it is understood today) at a point defined by six components $\sigma_{ij}$ and derived equations of motion and equilibrium. The Cauchy equations are currently accepted for isotropic materials. In 1828, A. L. Cauchy obtained Hooke’s law with 21 constants; under certain assumptions, however, the number of constants is reduced to 15 and even to 1 constant for an isotropic material. Equations similar to the Navier and Cauchy equations were also derived by S. D. Poisson (1828). There was a long discussion in papers dealing with the elasticity theory (see, e.g., [3, 26]) on the so-called multi-constant and rare-constant theories, i.e., on the number of independent constants in the generalized Hooke’s law (21 or 15 constants for arbitrary anisotropy and, correspondingly, 2 or 1 constant for an isotropic material). Experimental data, in particular, W. Voigt’s experiments aimed at studying elastic properties of crystals did not confirm that the six Cauchy conditions [3] $E_{ik lj} - E_{ik j} = 0$ were satisfied, i.e., that the number of independent elasticity moduli was 15. These conditions are approximately satisfied for beryl and rock salt. Only in the beginning of the 20th century, after the papers of M. Born on the crystal lattice theory were published [3, 33], it was finally recognized that Hooke’s law in the general case contains 21 constants.

A. L. Cauchy (1830) and G. Green (1839) studied propagation of plane waves in an elastic medium and derived equations for the propagation velocity as a function of the direction of the normal to the wave front [3]. In the general case, the wave surface [25] consists of three surfaces: for an isotropic medium, all three surfaces are spheres, and two of them coincide with each other [3].