Approximating the chromatic index of multigraphs

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Abstract It is well known that if \( G \) is a multigraph then \( \chi'(G) \geq \chi'^*(G) := \max\{\Delta(G), \Gamma(G)\} \), where \( \chi'(G) \) is the chromatic index of \( G \), \( \chi'^*(G) \) is the fractional chromatic index of \( G \), \( \Delta(G) \) is the maximum degree of \( G \), and \( \Gamma(G) = \max\{|E(G[U])|/(|U| − 1) : U \subseteq V(G), |U| \geq 3, |U| \text{ is odd}\} \). The conjecture that \( \chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\} \) was made independently by Goldberg (Discret. Anal. 23:3–7, 1973), Anderson (Math. Scand. 40:161–175, 1977), and Seymour (Proc. Lond. Math. Soc. 38:423–460, 1979). Using a probabilistic argument Kahn showed that for any \( c > 0 \) there exists \( D > 0 \) such that \( \chi'(G) \leq \chi'^*(G) + c\chi'^*(G) \) when \( \chi'^*(G) > D \). Nishizeki and Kashiwagi proved this conjecture for multigraphs \( G \) with \( \chi'(G) > \lfloor \frac{11\Delta(G) + 8}{10} \rfloor \); and Scheide recently improved this bound to \( \chi'(G) > \lfloor \frac{15\Delta(G) + 12}{14} \rfloor \). We prove this conjecture for multigraphs \( G \) with \( \chi'(G) > \lfloor \Delta(G) + \sqrt{\Delta(G)} \rfloor \), improving the above mentioned results. As a consequence, for multigraphs \( G \) with \( \chi'(G) > \Delta(G) + \sqrt{\Delta(G)} \) the answer to a 1964 problem of Vizing is in the affirmative.

Keywords Multigraph · Edge coloring · Chromatic index · Fractional chromatic index

DOI 10.1007/s10878-009-9232-y
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G. Chen is partially supported by NSF.
X. Yu is partially supported by NSA and by NSFC Project 10628102.
W. Zang is supported in part by the Research Grants Council of Hong Kong.

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1 Introduction

Let $G$ be a multigraph. A $k$-edge-coloring of $G$ is an assignment of $k$ colors to the edges of $G$ so that no two adjacent edges receive the same color. The chromatic index of $G$, denoted by $\chi'(G)$, is the smallest $k$ for which $G$ admits a $k$-edge-coloring. As it is $NP$-hard to determine $\chi'(G)$ (see Holyer 1980), a good estimate of $\chi'(G)$ has been the focus of extensive research.

Let $\Delta(G)$ denote the maximum degree of $G$. Clearly $\chi'(G) \geq \Delta(G)$. A classical theorem of Shannon (1949) asserts that $\chi'(G) \leq 3\Delta(G)/2$. Vizing (1964) and Gupta (1967) proved that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of an edge of $G$; and Kierstead (1984) studied the graphs $G$ for which $\chi'(G) = \Delta(G) + \mu(G)$. This Vizing–Gupta result implies that if $G$ is a simple graph then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Another lower bound for $\chi'(G)$ is the fractional chromatic index defined below. Let $\Gamma(G) = \max\left\{\frac{2|E(G[U])|}{|U| - 1} : U \subseteq V(G), |U| \geq 3, \text{ and } |U| \text{ is odd}\right\}$, where $G[U]$ is the subgraph of $G$ induced by $U$. Observe that if $U \subseteq V(G)$ and $|U|$ is odd, then every matching in $G[U]$ has size at most $(|U| - 1)/2$. Consequently $\chi'(G) \geq \Gamma(G)$; and hence $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$. The number $\chi^*(G) := \max\{\Delta(G), \Gamma(G)\}$ is the fractional chromatic index of $G$ (see Edmonds 1965; Schrijver 2003 and Seymour 1979), which is the solution to a linear program. The fractional chromatic index can be determined in polynomial time by using the ellipsoid algorithm; since the corresponding separation problem is equivalent to the weighted matching problem, see Theorem 28.6 in (Schrijver 2003).

In the 1970s, Goldberg (1973), Andersen (1977), and Seymour (1979) independently (and in different forms) made the following conjecture.

**Conjecture 1.1** For any multigraph $G$, $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\}$.

Note that Conjecture 1.1, if true, would imply that $\chi'(G) \leq 1 + \chi^*(G)$ for all multigraphs $G$. When studying conjectures of Tutte and Fulkerson about factorizations of cubic graphs, Seymour (1979) also made the following conjecture which is slightly weaker than Conjecture 1.1, but still achieves what Vizing’s theorem does for simple graphs.

**Conjecture 1.2** For any multigraph $G$, $\chi'(G) \leq 1 + \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$.

Conjecture 1.2 has an equivalent formulation in terms of $r$-graphs. Let $r$ be a positive integer. A multigraph $G = (V, E)$ is called an $r$-graph if $G$ is $r$-regular and, for every $X \subseteq V$ with $|X|$ odd, the number of edges between $X$ and $V - X$ is at least $r$ (in particular, $|V|$ is even). Seymour (1979) proved that Conjecture 1.2 is equivalent to the conjecture that if $G$ is an $r$-graph then $\chi'(G) \leq r + 1$.

Over the past three decades Conjecture 1.1 has been studied extensively, see, for instance (Goldberg 1984; Hochbaum et al. 1986; Marcotte 1986, 1990, 1990; Nishizeki and Kashiwagi 1990; Seymour 1990; Plantholt and Tipnis 1991; Kahn...