Generic Quasi-convergence for Strongly Order Preserving Semiflows: A New Approach*

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The principal result of the theory of monotone semiflows says that for an open and dense set of initial data, the trajectory converges to the set of equilibria. We show that strong compactness assumptions on the semiflow, required for the proof, can be replaced by the assumption that limit sets have infima and suprema in the space. This assumption is automatically satisfied in nice subsets of the space of continuous functions on a compact set and in euclidean space with respect to typical orderings.

KEY WORDS: Monotone dynamics; quasiconvergence.

1. INTRODUCTION

It is now well known that the generic pre-compact orbit of a strongly monotone dynamical system approaches the set of equilibria. Such a result is proved by Hirsch in [3] for ordinary differential equations where generic means that it holds for almost all initial data, relative to Lebesgue measure. This is extended to general strongly monotone systems in ordered spaces in [2,6] where generic refers either to a residual set of initial data or all but a subset of Gaussian measure zero in suitable Banach spaces. Matano [10] announced similar results. Smith and Thieme [15,16] and later Takáč [17] found mild conditions under which generic means the result holds for an open and dense set of initial data. Although these results hold for a quite general class of strongly ordered spaces, the proofs are not easy and, at least in the case that generic means open and dense,

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additional compactness assumptions on the semiflow are required. In the present paper, we give a short proof that convergence to the set of equilibria holds for an open and dense set of initial data. Of course, it is based on fundamental results of monotone systems theory, such as the Convergence Criterion and the Limit Set Dichotomy. However, instead of requiring additional compactness assumptions, we assume that compact invariant sets have a supremum and infimum in the state space. As the stronger property holds for the space of continuous functions on a compact set with the usual ordering (every compact subset has an infimum and supremum), our result covers the standard spaces used in applications to systems of ordinary and delay differential equations. For systems of reaction–diffusion equations, the choice of state space is more delicate but in many cases one can use continuous function spaces (see e.g. [13]). For an up-to-date survey of monotone systems theory, we refer the reader to our forthcoming paper [8]. For applications and related results, see [4,5,7,9,14].

2. DEFINITIONS AND BASIC RESULTS

Let $X$ be an ordered metric space with metric $d$ and partial order relation $\leq$. Recall that a partial order relation satisfies: (i) reflexive: $x \leq x$ for all $x \in X$; (ii) transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$; (iii) antisymmetric: $x \leq y$ and $y \leq z$ implies $x = y$. We write $x < y$ if $x \leq y$ and $x \neq y$. Given two subsets $A$ and $B$ of $X$, we write $A \leq B (A < B)$ when $x \leq y (x < y)$ holds for each choice of $x \in A$ and $y \in B$. We assume that the order relation and the topology on $X$ are compatible in the sense that $x \leq y$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all $n$. This is just to say that the partial order relation is closed. For $A \subset X$ we write $\bar{A}$ for the closure of $A$ and $\text{Int} A$ for the interior of $A$. A subset of an ordered space is unordered if it does not contain points $x, y$ such that $x < y$. Let $A \subset X$ and let $L = \{x \in X : x \leq A\}$ be the (possibly empty) set of lower bounds for $A$ in $X$. In the usual way, we define $\inf A := u$ if $u \in L$ and $L \leq u$; $u$ is unique if it exists. Similarly, $\sup A$ is defined.

The notation $x \ll y$ means that there are open neighborhoods $U, V$ of $x, y$ respectively such that $U \leq V$. Equivalently, $(x, y)$ belongs to the interior of the order relation. The relation $\ll$, sometimes referred to as the strong ordering, is transitive and vacuously antisymmetric, but not reflexive; in many cases it is empty. We write $x \geq y$ to mean $y \leq x$, and similarly for $>$ and $\gg$.

A semiflow on $X$ is a continuous map $\Phi : \mathbb{R}^+ \times X \to X, (t, x) \mapsto \Phi_t(x)$ such that:

$\Phi_0(x) = x, \quad (\Phi_t \circ \Phi_s)(x) = \Phi_{t+s}(x) \quad (t, s \geq 0, x \in X)$. 