A method is sketched to determine the presence of non-degenerate Hamiltonian Hopf bifurcations in three-degree-of-freedom systems by putting the bifurcation into standard form. Detailed computations are performed for the non-trivial example of the 3D Henon–Heiles family. After a careful formulation of the local once reduced system in terms of properly chosen invariants the system can be compared to the standard form by the application of singularity theoretic results.

**KEY WORDS:** Hamiltonian system; bifurcation; normal form; reduction; singularity; Henon-Heiles family; Hamiltonian Hopf bifurcation; relative equilibria.

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Examples also appear in buckling problems in the modeling of elastic struts \cite{7,8,18}. There are also examples in celestial mechanics. One of them is the restricted problem of three bodies, where the study of this bifurcation finds its origin \cite{25}. Hamiltonian Hopf bifurcations are also present in the 3D Hénon-Heiles family. In their original paper \cite{17} Hénon and Heiles considered the movement of a star in an axially symmetric galaxy. They considered planar motions, for which the vertical component of the angular momentum vanishes, and derived the famous model of a two-degree-of-freedom Hamiltonian system with a cubic potential. The 3D Hénon-Heiles family \cite{14,16} is the 3D generalization of this model, now allowing for non-zero values of the vertical component of the angular momentum as well and with a parameter introduced in the cubic term (see Eq. (8)). The 3D Hénon-Heiles family exhibits a complex bifurcational behavior. There are several Hamiltonian Hopf bifurcations which are different in nature (degenerate, non-degenerate, subcritical, and supercritical). It is for this reason that the 3D Hénon-Heiles family is taken as the leading problem in this paper.

The mechanism of the Hamiltonian Hopf bifurcation is best described on $\mathbb{R}^4$ with standard symplectic form. Let a family of Hamiltonian systems be given, depending on a parameter $\lambda$. Suppose that the origin is a stationary point and that for some value $\lambda_0$ the linearized system consists of two harmonic oscillators in 1:1 resonance. Then for $\lambda_0$ the system linearized at the origin has a pair of purely imaginary eigenvalues $\pm i\nu_0$ and generically the eigenvalues will behave as follows when the parameter $\lambda$ passes through $\lambda_0$: four purely imaginary eigenvalues, moving in pairs $\pm i\nu_1$ and $\pm i\nu_2$, will meet for $\lambda = \lambda_0$ at $\pm i\nu_0$ and move off from the imaginary axis into the complex plane to form a complex quartet $\pm \alpha \pm i\beta$. This bifurcation in two-degree-of freedom systems is known as the Hamiltonian Hopf bifurcation \cite{25}, which name was chosen because it is the Hamiltonian equivalent of the non-conservative (linear) Hopf bifurcation where a pair of eigenvalues crosses the imaginary axis. The eigenvalue movement in the Hamiltonian case is sometimes also called a Krein collision, cf. \cite{20,22}. Like in the non-conservative equivalent the equilibrium at the origin changes from stable to unstable. Whether this bifurcation is non-degenerate, and hence of co-dimension one, has to be decided by considering the higher order terms of the Hamiltonian. To this end one normalizes with respect to the linearized part, which allows to transform away the third order terms of the Hamiltonian. The non-degeneracy condition then requires that a certain coefficient of the (normalized) fourth order terms be non-zero. See \cite{16} for a geometric criterion for this non-degeneracy condition and \cite{27} for further transversality conditions.