Invertible Contractions and Asymptotically Stable ODE’S that are not $C^1$-Linearizable

Hildebrando M. Rodrigues$^1$ and J. Solà-Morales$^{2,3}$

Received December 3, 2005

We present an example of a contraction diffeomorphism in infinite dimensions that is not $C^1$-linearizable, and we construct a regular ordinary differential equation in a Hilbert space whose time-one map is that diffeomorphism. With this we have an example of an asymptotically stable ODE that is not $C^1$-conjugate to its linear part.

KEY WORDS: linearization; conjugacy; contraction.

AMS CLASSIFICATION: PRIMARY: 37L10; 34G20, SECONDARY: 35B40; 37C15.

1. INTRODUCTION AND MAIN RESULT

There is a classical result due to Hartman, published in [6], that asserts that any smooth contraction diffeomorphism in finite dimensions can be transformed, in a neighborhood of its fixed point, into its linear part with a transformation that is a diffeomorphism of class $C^1$. Of course, the theorem also applies to systems of ordinary differential equations, in finite dimensions. This result has been extended by several authors to cover some situations in infinite dimensional Banach spaces.

These extensions to infinite dimensions started with the works by Mora and Sola-Morales [8] and Bin Tan [13], and have been continued in more recent times by the independent works of Bialy [3], Rodrigues

---

$^1$Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil. E-mail: hmr@icmc.usp.br

$^2$Departament de Matemática Aplicada 1, Universitat Politècnica de Catalunya, Av. Diagonal 647, 08028 Barcelona, Spain. E-mail: jc.sola-morales@upc.edu

$^3$To whom correspondence should be addressed.
Theorem 1. (Linearization for contractions.) Let $Z$ be a Banach space with the property that there exists function $\rho$ such that

$$\rho \in C^1_1(Z, \mathbb{R}) \text{ with } \rho(z) = 1 \text{ when } |z| \leq 1/2 \text{ and } \rho(z) = 0 \text{ when } |z| \geq 1. \quad (1.1)$$

Suppose that $L, L^{-1} \in \mathcal{L}(Z)$. We assume that there exist real numbers $v_i^-, v_i^+, i = 1, \ldots, n$ such that:

$$0 < v_n^- < v_n^+ < v_{n-1}^- < \cdots < v_1^- < v_1^+ < 1$$

$$v_i^+ v_i^- < v_i^-, \quad i = 1, \ldots, n \quad \text{(non-resonance condition)} \quad (1.2)$$

$$|\sigma(L)| \subset (v_n^-, v_n^+) \cup (v_{n-1}^-, v_{n-1}^+) \cup \cdots \cup (v_1^-, v_1^+).$$

Let $F = F(z)$ be a $C^{1,1}$-function in a neighborhood of the origin with values in $Z$, such that $F = 0, \partial_z F = 0, \text{ at } z = 0$.

Then, for the map $T: z \mapsto z', z' = Lz + F(z)$, there exists a $C^{1}$-map $R: z \mapsto u, u = z + \psi(z)$, satisfying $\psi = 0, \partial_z \psi = 0, \text{ at } z = 0$, such that $R T^{-1}: u \mapsto u'$ has the form $u' = Lu$ in a sufficiently small neighborhood of the origin.

It is worth to say that one of the main natural applications of this theorem, and of similar results, is to partial differential equations of the type of the damped wave equation (see [8, 10]).

Also, we want to point out that some work on $C^1$-linearization for non-contraction diffeomorphisms in infinite dimensions has also been done: in [11], we proved this type of result when the linear part is of a quite particular saddle case. This result in fact extends a result proved by Hartman (see [6]) for two-dimensional systems (see also [2]).

All of the results mentioned above were in the positive direction, that is, in the direction of showing that a smooth linearization exists provided that the linear part satisfies some hypotheses. But until quite recently it was not known, for example, whether or not every smooth contraction in infinite dimensions can be locally conjugated to its linear part in the class $C^1$. This was also explicitly mentioned as an open problem in the Remark 2 of Abacci [1]. Then, in a short Note, Rodrigues and Sola-Morales [12], we presented a first example of an analytic invertible contraction that is not $C^1$-linearizable. Of course, the non-resonance condition of (1.2) is not satisfied in the example, but (1.1) holds.