Chaos and Entropy for Interval Maps

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Abstract In this paper, various chaotic properties and their relationships for interval maps are discussed. It is shown that the proximal relation is an equivalence relation for any zero entropy interval map. The structure of the set of $f$-nonseparable pairs is well demonstrated and so is its relationship to Li-Yorke chaos. For a zero entropy interval map, it is shown that a pair is a sequence entropy pair if and only if it is $f$-nonseparable. Moreover, some equivalent conditions of positive entropy which relate to the number “3” are obtained. It is shown that for an interval map if it is topological null, then the pattern entropy of every open cover is of polynomial order, answering a question by Huang and Ye when the space is the closed unit interval.

Keywords Chaos · Entropy · Interval map

Mathematics Subject Classification (2000) 37E05 · 37B40 · 54H20

1 Introduction

The study of the complexity or chaotic behavior is a central topic in topological dynamics. Starting from the work of Li and Yorke [18] various authors introduce a lot of definitions of chaos according to their understanding of the phenomena. Among them, Li-Yorke chaos, Denavey chaos [7] and positive entropy [6] are popular ones. It is important to understand their relationships. Recently, it has been shown that for a general topological dynamical system, Devaney chaos implies Li–Yorke chaos [11] and positive entropy implies Li–Yorke chaos [4].
In the study of the so called “local entropy theory” (for a survey see [10]), a lots of notions are introduced to describe dynamical properties. It is not clear the relationship of those properties (related to entropy) with the chaotic behaviors for a given space. The purpose of the current paper is to study the relationship in the case when the given space is a closed interval. We believe that many results of the paper hold for a graph map even more general spaces.

To state our results, we introduce some notations first. Let \( I \) be the closed unit interval \([0, 1]\) and \( C(I, I) \) denote the class of continuous maps of \( I \) to itself. For \( f \in C(I, I) \), let \( f^0 \) be the identity, and for \( n \in \mathbb{N} \), let \( f^{n+1} = f^n \circ f \), where \( \mathbb{N} \) stands for the set of positive integers.

A point \( x \in I \) is called a periodic point of \( f \) with period \( n \) if \( f^n(x) = x \), \( f^k(x) \neq x \) for \( 1 \leq k < n \). A periodic point with period 1 is called a fixed point. The \( \omega \)-limit set of \( x \), denoted by \( \omega_f(x) \), is the set of limit points of \( \{f^i(x)\}_{i=0}^{\infty} \). A set \( W \subset I \) is called some \( \omega \)-limit set for \( f \), if there exists an \( x \in I \) such that \( W = \omega_f(x) \). Denote the collection of all \( \omega \)-limit sets for \( f \) by \( \omega_f \).

A point \( x \in X \) is called (1) a recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( n > 0 \), such that \( f^n(x) \in U \); (2) a strongly recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( N > 0 \), such that if \( f^n(x) \in U \) then \( f^{m+k}(x) \in U \) for some \( k \) with \( 0 < k \leq N \); (3) a regularly recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( N > 0 \), such that \( f^{kN}(x) \in U \) for all \( k > 0 \).

Denote \( \text{Per}(f), \text{Rec}(f), \text{SR}(f) \) and \( \text{RR}(f) \) by the set of periodic points, recurrent points, strongly recurrent points and regularly recurrent points, respectively. It is well known that

\[
\text{Per}(f) \subset \text{RR}(f) \subset \text{SR}(f) \subset \text{Rec}(f).
\]

The terminology “chaos” was first introduced by Li and Yorke [18] to describe the complex behavior of trajectories. A pair \( \langle x, y \rangle \in I^2 \) is called proximal if \( \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0 \) and is called asymptotic if \( \lim_{n \to \infty} |f^n(x) - f^n(y)| = 0 \). A scrambled pair or Li–Yorke pair is one that is proximal but not asymptotic. A pair \( \langle x, y \rangle \) is called proper if \( x \neq y \).

For \( \delta > 0 \), a pair \( \langle x, y \rangle \) is said to be \( \delta \)-scrambled if

\[
\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0 \quad \text{and} \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| \geq \delta.
\]

A set \( C \subset I \) is called scrambled (resp. \( \delta \)-scrambled) if any proper pair \( \langle x, y \rangle \in C^2 \) is scrambled (resp. \( \delta \)-scrambled). The map \( f \) is called Li–Yorke chaotic (resp. \( \delta \)-Li–Yorke chaotic) if there exists an uncountable scrambled set (resp. \( \delta \)-scrambled set).

In [18], Li and Yorke proved that for an interval map period 3 implies Li–Yorke chaos. In [14], Jankova and Smítal generalized this result as follows: if an interval map has positive entropy, then it is Li–Yorke chaotic.

The converse of this result is not true: Xiong [26] and Smítal [24] constructed some interval maps with zero entropy which are Li–Yorke chaotic.

In [24], Smítal also built some useful tools for zero entropy interval maps: the periodic portion of an \( \omega \)-limit set and \( f \)-nonseparable points. See [6] for another approach to the periodic portion of an \( \omega \)-limit set.

In this paper, we discuss those various chaotic properties and their relationships for interval maps. In Sect. 2, for preparation we recall some basic definitions and results for a general dynamical system. In Sect. 3, we review the structure of the \( \omega \)-limit set and build a new approach to the periodic portion of an \( \omega \)-limit set.

In Sect. 4, we deal with zero entropy interval maps. First, we show that the proximal relation is an equivalence relation for a zero entropy interval map. Second, some properties