A Shapley-based decomposition of the $R$-Square of a linear regression

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Abstract This note suggests a new way of determining the exact contributions of the explanatory variables to the $R$-Square of a linear regression. The proposed methodology combines the so-called Shapley approach (Chantreuil and Trannoy, Inequality decomposition values: the trade-off between marginality and consistency. THEMA Discussion Paper, Université de Cergy-Pontoise, France 1999; Shorrocks, Decomposition Procedures for Distributional Analysis: A Unified Framework Based on the Shapley Value (mimeo), University of Essex, 1999) with the Fields (Res. Labor Econ., 22:1–38, 2003) decomposition.

Key words inequality decomposition · $R$-Square · Shapley value.

JEL Classification C20 · D63

1 Introduction

The determination of the exact contributions and statistical significance of each explanatory variable to the variance of the dependent variable of a regression is an important issue in applied economics. The most common approach to this problem is to use stepwise regression procedures and significance tests. However, the stepwise regression method is of an arbitrary nature, while significance tests do not always allow the ranking of the explanatory variables in order of importance. A systematic way of quantifying the different contributions of the explanatory variables to the goodness of fit of a regression should therefore, without any doubt, be of use in many fields of research.

In a recent paper, Fields [5] showed how the decomposition technique that breaks down income inequality into income sources could be applied to linear regression analysis to estimate the contributions of explanatory variables to the dispersion of the dependent variable.1

1Decomposing income inequality by factor components was analyzed extensively in the literature [e.g., 4, 7, 9]. Overall income inequality was also decomposed by population subgroups [e.g., 1, 10, 12].

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An interesting alternative approach to the decomposition of inequality was recently suggested by Chantreuil and Trannoy [2] and Shorrocks [11]. They used the concept of Shapley value in cooperative game theory to determine the exact contribution of different income sources to overall inequality.

In this note, this so-called “Shapley approach” is developed to derive the exact contributions of the various explanatory variables of a linear regression to its $R^2$. Remembering that the contributions of the different explanatory variables to the $R^2$ are actually the percentage contributions of these variables to the variance of the dependent variable, it should be clear that in the case of Mincerian earnings function this new decomposition will allow us deriving the contribution of various determinants to the overall wage inequality. The new Shapley decomposition is then compared with that of Fields [5].

The Shapley procedure, however, is a technique that may be applied in several ways, leading to different results. Particularly, Shorrocks [11] outlined another way of using the Shapley procedure for decomposing wage dispersion. Following Shorrocks’ ideas, we shortly show that this application of the Shapley procedure actually leads to the Fields [5] decomposition.

While the Fields decomposition may identify the effects of the explanatory variables in simple linear regressions, the new approach may also be applied to more complicated ones, like regressions including interactions or dummy variables. In the case of high multicollinearity between explanatory variables, there are important differences between our decomposition and that suggested by Fields, and the decomposition suggested in this note seems to have clear advantages. An empirical illustration based on Israeli data seems to confirm the attractiveness of the new methodology that has been proposed in this note.

2 Decomposing the $R^2$

Assume a regression expressed as:

$$y = a + \sum_{j=1}^{J} b_jx_j + e.$$  \hspace{1cm} (1)

The variance of the dependent variable, i.e., the total sum of squares (TSS), may be decomposed into the regression sum of squares (RSS), meaning the deviations from the mean explained by the regression, plus the error sum of squares (ESS), the unexplained deviations from the regression line:

$$\text{Var}(y) = TSS = \text{Var}(\hat{y}) + \text{Var}(e) = RSS + ESS,$$  \hspace{1cm} (2)

where $\hat{y}$ is the predicted value of the dependent variable.

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2 Further applications are given by Sastre and Trannoy [8].

3 The game-theory related concepts used in this note (“a Shapley decomposition” etc.) do not follow the true definition of these concepts as originally used in games theory. We nevertheless still use these terms to describe the decomposition procedures applied here, since this note follows previous papers [2, 11] in which these terms were introduced.

4 I want to thank an anonymous referee who suggested following Shorrocks’ marginal decomposition. As explained later in the note, this procedure decomposes the variance of the dependent variable (and not the explained variance, as we suggest above) using the base (and full) regression model.