A dual algorithm for the minimum covering weighted ball problem in \( \mathbb{R}^n \)

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Abstract The nonlinear convex programming problem of finding the minimum covering weighted ball of a given finite set of points in \( \mathbb{R}^n \) is solved by generating a finite sequence of subsets of the points and by finding the minimum covering weighted ball of each subset in the sequence until all points are covered. Each subset has at most \( n + 1 \) points and is affinely independent. The radii of the covering weighted balls are strictly increasing. The minimum covering weighted ball of each subset is found by using a directional search along either a ray or a circular arc, starting at the solution to the previous subset. The step size is computed explicitly at each iteration.

Keywords Minimum covering ball · Min-max location · One-center location · Nonlinear programming

1 Introduction

Given a set \( P = \{ p_1, \ldots, p_m \} \) of \( m \) distinct points in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), and a positive weight \( w_i > 0 \) for each point \( p_i \in P \), the minimum covering weighted ball problem is to determine the ball with center \( x \) and smallest radius \( z \) so that the weighted Euclidean distance between \( x \) and \( p_i \) is less than or equal to \( z \) for all \( p_i \in P \). The weighted Euclidean distance between \( x \) and \( p_i \) is denoted by \( w_i \| p_i - x \| \), and the problem, denoted by \( M(P) \), is written as:

\[
M(P) : \min_{x \in \mathbb{R}^n} \ max_{p_i \in P} w_i \| p_i - x \|.
\]

Problem \( M(P) \) has the following equivalent representation:

\[
M(P) : \min_{x \in \mathbb{R}^n} \ max_{p_i \in P} w_i \| p_i - x \|.
\]
This version of the problem is observed to be convex and is known as the weighted min-max location problem and as the weighted one-center location problem where the point $x$ is the location of a facility that minimizes the maximum weighted Euclidean distance between $x$ and any of the points $p_i$. If all the weights are equal, or equivalently, if all the weights are 1, the problem is called the minimum covering ball problem. This paper extends the dual algorithm of Dearing and Zeck [1] for the minimum covering ball problem to the minimum covering weighted ball problem.

2 Literature

Most of the literature concerns the minimum covering ball problem. In $\mathbb{R}^2$, the problem of finding the minimum covering circle was first posed by Sylvester [2] in 1857, and has been solved by several approaches. A geometric primal approach, developed by Sylvester [3] and Crystal [4] begins with a large circle covering all points in $P$, and each iteration alternates between reducing the radius and moving the center, without violating the covering property, until optimality is reached. Elzinga and Hearn [5] developed a geometric dual approach for the minimum covering circle in $\mathbb{R}^2$ by constructing the minimum covering circle of subsets $S \subseteq P$ in a sequence, with each subset containing at most three points and with the circle radii increasing until some circle covers the entire set $P$. Voronoi diagrams [6] have also been used to solve the unweighted problem in $\mathbb{R}^2$, and Meggido [7] developed a linear time algorithm for solving the problem.


For the minimum covering weighted ball problem in $\mathbb{R}^2$, Hearn and Vijay [18] present a geometrical dual type algorithm. Meggido [19] extends the theoretical linear time algorithm to the weighted case.

The minimum covering ball problem has also been considered for other than the Euclidean distance. A survey can be found in Plastria [20].

3 Approach

It is well known (e.g. Hearn and Vijay [18]) that a solution to the minimum covering weighted ball problem exists, that it is unique, that it lies in the convex hull of the points $p_i$, $i = 1, \ldots, m$, and that the solution is determined by at most $n + 1$ of the given points. It is also well known that the optimal solution $x$ must lie on at least one bisector of a pair of points, where a bisector of $p_i$ and $p_j$ is the set of all points of equal weighted distance to $p_i$ and $p_j$. These properties are used to develop the dual algorithm presented here.