Recent Developments in Understanding
Two-dimensional Turbulence and the Nastrom–Gage Spectrum

Eleftherios Gkioulekas and Ka-Kit Tung

Department of Applied Mathematics, University of Washington, Seattle, WA, USA

Two-dimensional turbulence appears to be a more formidable problem than three-dimensional turbulence despite the numerical advantage of working with one less dimension. In the present paper we review recent numerical investigations of the phenomenology of two-dimensional turbulence as well as recent theoretical breakthroughs by various leading researchers. We also review efforts to reconcile the observed energy spectrum of the atmosphere (the Nastrom–Gage spectrum) with the predictions of two-dimensional turbulence and quasigeostrophic turbulence.

PACS numbers: 42.68.Bz, 47.27.-i, 47.27.ek, 92.60.hk, 92.10.ak

1. INTRODUCTION

Turbulence is ubiquitous in the fluid environment we live in, and yet a fundamental theoretical understanding from first principles is not yet available, although considerable progress has been made in the case of isotropic and homogeneous three-dimensional turbulence. Large-scale flows in thin fluid shells, such as planetary atmospheres and the ocean, tend to be quasi-two-dimensional. Two-dimensional flows differ from three-dimensional turbulence in that there are usually two closely related conservative quantities exchanged by nonlinear triad interactions. Furthermore, the cascades of two-dimensional turbulence do not exhibit universal behavior with the same degree of consistency that we have come to expect from three-dimensional turbulence. Also interesting is the inverse energy cascade, unique in “2d-like” systems, where an initially noisy velocity field continuously forced by white noise small-scale forcing will nonetheless evolve into coherent vortical structures. The striking resemblance between the pattern formation of two-dimensional turbulence and similar patterns
in the atmospheres of gas-giant planets, like Jupiter, tickles the imagination and raises interesting but hard questions.\textsuperscript{1}

When Kraichnan,\textsuperscript{2} Leith\textsuperscript{3} and Batchelor\textsuperscript{4} first pioneered the study of two-dimensional turbulence, it was thought that it would be easier to handle theoretically and simpler to simulate numerically than three-dimensional turbulence. The fact that no convincing simulation of the dual cascades predicted by KLB, with an upscale energy cascade and a downscale enstrophy cascade, has been achieved during the ensuing three decades is a hint that the problem of two-dimensional turbulence is richer than was thought, perhaps even richer than the three-dimensional isotropic homogeneous turbulence. In addition, because geophysical fluids behave more like two-dimensional fluids than three-dimensional isotropic homogeneous fluids, it is not possible to simply ignore the theoretical and numerical problems of two-dimensional turbulence on the grounds that it is a fictitious fluid.

In the present paper, we shall review some of the recent breakthroughs in understanding two-dimensional turbulence. We shall also review the problem of the Nastrom–Gage energy spectrum of the atmosphere, and recent theories that have been proposed to explain it. Needless to say, this review is biased to reflect the viewpoint and interests of the authors. Less biased reviews of two-dimensional turbulence\textsuperscript{5–7} and quasi-geostrophic turbulence\textsuperscript{8,9} are available in the literature. Good reviews on the Nastrom–Gage spectrum can also be found in the papers by Lindborg.\textsuperscript{10,11}

This paper is organized as follows. Sections 2 and 3 of the paper deal with two-dimensional turbulence. Section 4 discusses the problem of the Nastrom–Gage energy spectrum. Finally, Section 5 reviews the method of spectral reduction, because we believe that it has the potential to lead to further breakthroughs in this field.

\section{Dynamics of Two-Dimensional Turbulence}

Let $u_\alpha(r,t)$ be the Eulerian velocity field. The governing equations of two-dimensional turbulence are:

\begin{align}
\frac{\partial u_\alpha}{\partial t} + u_\beta \partial_\beta u_\alpha &= -\partial_\alpha p + \mathcal{D}u_\alpha + f_\alpha, \tag{1} \\
\partial_\alpha u_\alpha &= 0, \tag{2}
\end{align}