On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order (with applications to the construction of the associated Legendre function of the second kind of integer degree and order)

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Abstract  The derivative of the associated Legendre function of the first kind of integer degree with respect to its order, \( \partial P_\mu^n(z) / \partial \mu \), is studied. After deriving and investigating general formulas for \( \mu \) arbitrary complex, a detailed discussion of \( [\partial P_\mu^n(z) / \partial \mu]_{\mu = \pm m} \), where \( m \) is a non-negative integer, is carried out. The results are applied to obtain several explicit expressions for the associated Legendre function of the second kind of integer degree and order, \( Q_{\pm m}^n(z) \). In particular, we arrive at formulas which generalize to the case of \( Q_{m}^n(z) \) (0 ≤ \( m \) ≤ \( n \)) the well-known Christoffel’s representation of the Legendre function of the second kind, \( Q_n(z) \). The derivatives \( [\partial^2 P_\mu^n(z) / \partial \mu^2]_{\mu = m} \), \( [\partial Q_\mu^n(z) / \partial \mu]_{\mu = m} \) and \( [\partial Q_{\mu - 1}^n(z) / \partial \mu]_{\mu = m} \), all with \( m > n \), are also evaluated.

Keywords  Legendre functions · Parameter derivative · Special functions

1 Introduction

It is the purpose of the present paper to contribute to the theory of special functions of mathematical physics and chemistry. Specifically, we shall add to the knowledge about the associated Legendre functions of the first, \( P_\nu^\mu(z) \), and second, \( Q_\nu^\mu(z) \), kinds (cf, e.g., [1–29]).\(^1\) We shall touch two particular problems. First, we shall investigate

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\(^1\) The associated Legendre function of the second kind defined by Barnes [10] differs from the counterpart function of Hobson [12] used in the present paper. The relationship between the two functions is: \( [Q_\nu^\mu(z)]_{\text{Barnes}} = [e^{-ix\mu} \sin(\pi(\nu + \mu))/\sin(\pi \nu)] Q_\nu^\mu(z) \).
the derivative of the associated Legendre function of the first kind of integer degree with respect to its order. Second, it will be shown that the results of that investigation may be used to construct, in a straightforward and unified manner, several known representations of the associated Legendre functions of the second kind of integer degree and order, \( Q_n^m(z) \); in the past those representations were obtained by other authors with the use of a variety of, usually more complicated, techniques. In addition, some possibly new expressions for \( Q_n^m(z) \) will be also presented. They are potentially useful because the function \( Q_n^m(z) \) is encountered in solutions of numerous boundary-value problems of applied mathematics, e.g., in electro- and magnetostatics or in the theories of diffusion and heat conduction in spherical, spheroidal (both prolate and oblate), and conical geometries [15]. The function \( Q_n^m(z) \) appears also in the Neumann expansion [30] (cf also [15, Eq. 10.3.53]) for the inverse distance between two points. This expansion is exploited in quantum molecular physics, in studies of bound [31–37] and scattering [38,39] states of diatomics to handle the interelectronic Coulomb repulsion term in the prolate spheroidal coordinates.

The literature concerning the derivative \( \frac{\partial P_\mu^\nu(z)}{\partial \mu} \) is very limited. Surprisingly, no relevant expressions have been given in [29], which otherwise contains a large collection of parameter derivatives of various special functions. Several variants of the formula

\[
\left. \frac{\partial P_\mu^\nu(z)}{\partial \mu} \right|_{\mu=0} = \psi(\nu + 1) P_\nu(z) + Q_\nu(z),
\]

where

\[
\psi(\xi) = \frac{1}{\Gamma(\xi)} \frac{d\Gamma(\xi)}{d\xi}
\]

is the digamma function [14,20,23,25,40], may be found in [23, p. 178]. Robin [18, Eq.333, p. 175] gave the following representation\(^2\) of \( \frac{\partial P_\mu^\nu(z)}{\partial \mu} \):

\[
\frac{\partial P_\mu^\nu(z)}{\partial \mu} = \frac{1}{2} P_\mu^\nu(z) \ln \frac{z + 1}{z - 1} + \left( \frac{z + 1}{z - 1} \right)^{\mu/2} \sum_{k=0}^{\infty} \frac{\Gamma(\nu + k + 1) \psi(k - \mu + 1)}{k! \Gamma(\nu + k + 1) \Gamma(k - \mu + 1)} \left( \frac{z - 1}{2} \right)^k
\]

\(|z - 1| < 2\).

\(^2\) The notation used in the present work differs in some respects from that adopted by Robin in [17–19]. In particular, the digamma function used by Robin was defined as

\[
\psi(\xi) = \frac{1}{\Gamma(\xi + 1)} \frac{d\Gamma(\xi + 1)}{d\xi}
\]

rather than as in our Eq. 1.2. Also, one should be warned that in Eq. (333) on p. 175 in [18] the factor \( p! \) in the denominator of a summand is missing.