Level Set Equations on Surfaces via the Closest Point Method

Colin B. Macdonald · Steven J. Ruuth

Received: 30 May 2007 / Revised: 6 February 2008 / Accepted: 8 February 2008 / Published online: 11 March 2008
© Springer Science+Business Media, LLC 2008

Abstract Level set methods have been used in a great number of applications in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) and it is natural to consider extending some of these methods to problems defined on surfaces embedded in \( \mathbb{R}^3 \) or higher dimensions. In this paper we consider the treatment of level set equations on surfaces via a recent technique for solving partial differential equations (PDEs) on surfaces, the Closest Point Method (Ruuth and Merriman, J. Comput. Phys. 227(3):1943–1961, 2008). Our main modification is to introduce a Weighted Essentially Non-Oscillatory (WENO) interpolation step into the Closest Point Method. This, in combination with standard WENO for Hamilton–Jacobi equations, gives high-order results (up to fifth-order) on a variety of smooth test problems including passive transport, normal flow and redistancing. The algorithms we propose are straightforward modifications of standard codes, are carried out in the embedding space in a well-defined band around the surface and retain the robustness of the level set method with respect to the self-intersection of interfaces. Numerous examples are provided to illustrate the flexibility of the method with respect to geometry.

Keywords Closest Point Method · Level set methods · Partial differential equations · Implicit surfaces · WENO schemes · WENO interpolation

1 Introduction

The level set method [15] has been successfully applied to a tremendous variety of problems involving curve evolution in \( \mathbb{R}^2 \) or surface evolution in \( \mathbb{R}^3 \). This curve or surface—the interface—is represented as the zero contour of a level set function \( \phi \). A principal strength of
the level set method comes from its ability to handle changes in topology of the evolving interface, i.e., interfaces break apart or merge naturally, and without the need for special code or instructions to detect or treat the shape changes as they occur. A second, and also key, benefit that occurs when using level set methods is that the discretization of the underlying level set equation (of Hamilton–Jacobi type) can be carried out using well-known, accurate and reliable discretization techniques, such as the weighted essentially non-oscillatory (WENO) methods described in [7, 8, 10]. Taken together, these benefits have contributed to a widespread adoption of level set techniques in different disciplines [14, 21].

Level set methods have primarily been used to treat evolving interfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. It is natural to want to evolve level set equations on general domains, to give a way of robustly capturing the motion of interfaces on curved surfaces. Such an extension would be extremely interesting since it could open up the possibility of generalizing existing level set applications to curved surfaces. For example, suppose one wished to segment out objects appearing on a surface. By extending level set methods to surfaces, we gain the possibility of solving this problem by simply transferring existing level set methods for segmentation to the case of surfaces. This approach becomes even more compelling if the algorithms for surface flows end up being based on existing codes for standard two- and three-dimensional flows. Indeed, we shall see this is the case with the Closest Point Method.

An interesting method for evolving interfaces on surfaces was proposed by Cheng et al. [3]. In their approach, a level set representation of the underlying surface was taken, with the evolving interface being represented by the intersection of two level set functions. The level set evolution equation for $\phi$ made use of standard gradients followed by projections to the surface rather than using the surface gradients that would otherwise appear in a surface PDE. Thus, the method evolved a level set PDE in $\mathbb{R}^3$, and, at any time, gave the position of the interface on the surface as the zero contour of $\phi$ on the surface. See [3] for further details on the method as well as a fascinating selection of examples using the method.

An alternative way of developing a method to evolve interfaces on surfaces is to start from a level set equation defined on a surface, e.g., a Hamilton–Jacobi equation of the form

$$
\phi_t + H(t, x, \phi, \nabla \phi) = 0,
$$

$$
\phi(0, x) = \phi_0(x),
$$

or some curvature-dependent generalization of this, and to solve it with some existing strategy for evolving PDEs on surfaces. For example, one might apply the method of Bertalmío et al. [1] or Greer [6] to treat the surface PDE. These methods use a level set representation of the surface and replace surface gradients by standard gradients and projection operators in $\mathbb{R}^3$ to get an embedding PDE which is defined throughout time and space and agrees with the surface evolution on the surface. This leads to similar or the same PDEs as those appearing in [3], and will therefore be very similar in character to the methods described there.

In this paper, we will evolve the level set equations of Hamilton–Jacobi type (1) according to the recently proposed Closest Point Method [18]. The Closest Point Method has a number of properties that make it quite attractive for solving level set equations on surfaces. First of all, it takes the underlying surface representation to be a closest point representation. This allows it to treat PDEs on surfaces that have boundaries, lack any clearly defined inside/outside or are of arbitrary codimension. Similar to level set based methods, the method uses an embedding PDE defined in the embedding space (e.g., $\mathbb{R}^3$). The meaning and use of the embedding PDE is fundamentally different, however, since it is only valid initially, and therefore requires an extension step to ensure its accuracy. A desirable property of the...