ANALYSIS OF EXPLICIT AND IMPLICIT ASSUMPTIONS IN THE THEOREMS OF J. VON NEUMANN AND J. BELL

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Abstract

This paper is devoted to a detailed analysis of probabilistic assumptions in the theorems of J. von Neumann and J. Bell. We show that without a rigorous formalization of the probabilistic content of Bell’s arguments one cannot forcefully derive the fundamental dilemma that we are often being offered, that is, either nonlocality or the death of reality.

Keywords: theorems of J. von Neumann and J. Bell, nonlocality, death of reality, single Kolmogorov space, range of values, contextuality.

1. Introduction

Recent years have witnessed spectacular developments in quantum information theory, both within the theoretical and experimental frameworks (especially quantum optics); see e.g. [1]. In this process, it became more evident that some problems of foundations of quantum mechanics (which were of purely theoretical interest for a long time) became extremely important in experimental research and even in technology (see, e.g., [2–29] on recent studies of quantum foundations). One of such fundamental problems is the problem of completeness of quantum mechanics, roughly speaking: Can one go beyond quantum mechanics? In the first stage of the debate between Einstein and Bohr, this problem had a merely theoretical value. However, nowadays, e.g., the claims on the extremely high security level of quantum cryptographic protocols are essentially based on the assumption of completeness of quantum mechanics.

There is a rather common opinion that quantum mechanics is complete. The $\psi$-function provides the most complete description of a physical system. It is impossible to create models providing a better description than given by quantum mechanics (impossibility of introducing the so-called “hidden variables”). The only alternative to completeness is nonlocality. However, it is not easy to proceed consistently in the realistic but nonlocal framework (although it is, in principle, possible, as in Bohmian mechanics). Moreover, in the realistic nonlocal framework, one could not guarantee the security of quantum cryptography, see [29]. Even if one speaks, in general, about the contradiction between local realism and quantum mechanics, in applications he uses completeness of quantum mechanics.

NO-GO theorems (impossibility theorems) are crucial arguments in the debate on completeness of quantum mechanics. There is a number of different NO-GO theorems of von Neumann [30], Bell [31], Cochen–Specker [32], and so on. From the very beginning, we point out that it is a little bit surprising...
that there are so many NO-GO theorems. It would be better to have just one NO-GO theorem, which
would imply completeness of quantum mechanics. The situation where a new theorem “improves” a
former theorem is a little bit disturbing. It could be considered as a sign that there was something wrong
with the previous NO-GO theorem.

In this paper, we would like to analyze carefully the explicit and implicit assumptions of the NO-GO
theorems of J. von Neumann and J. Bell. The main attention is paid to Bell’s theorem.

I shared the common viewpoint with a number of authors — one that I have been trying to advocate
that without a rigorous formalization of the probabilistic content of Bell’s arguments one cannot forcefully
derive the fundamental dilemma that we are often being offered, that is, either nonlocality or the death
of reality [33–43].

For example, if one uses the Kolmogorov measure-theoretic model (as J. Bell did) then one should
be aware that there is no reason, even in classical physics, to assume that statistical data obtained in
different experiments should be described by a single Kolmogorov probability space; see, e.g., [21] for
details. In particular, K. Hess and W. Philipp [27] found a theorem (proved by a Soviet mathematician
N. N. Vorob’ev [44]) describing the conditions that are necessary and sufficient for the realization of a
few random variables on a single Kolmogorov space. We remark that Vorob’ev’s theorem
was proved before the inequality that is nowadays known as Bell’s inequality appeared in quantum physics. Vorob’ev
applied his results to game theory.

For physics, the crucial point is that statistical data inducing nonclassical probabilistic consequences,
e.g., violation of Bell’s inequality, is sampled in a few different experiments, which are determined by
different experimental settings, e.g., orientations of polarization beam splitters or Stern–Gerlach magnets,
or different numbers of open slits. Therefore, although the assumption about a single Kolmogorov space
is so attractive from the mathematical viewpoint (in particular, because it simplifies all considerations),
it is not so much justified from the experimental viewpoint. It is interesting that some experimentalists
working in quantum optics understood this purely mathematical problem in Bell’s arguments (see
Klyshko [45–50] and, especially, [51], p. 976): “Nonclassicality of quantum physics is, in fact, the impossibility of introducing the joint probability distributions for noncommutative operators…”

Moreover, in physics literature it was often pointed out that in Bell’s framework one does not need to
use the Kolmogorov measure-theoretic model at all, because it is possible to operate just with frequencies.
As was remarked first by De Baere [52] and then by Khrennikov [21], the frequency derivation of Bell’s
inequality is also based on mixing of statistical data from different experiments. Such a procedure was
not totally justified. If we proceed in a rigorous mathematical way, by using the von Mises frequency
approach (recently presented on the mathematical level in [21]), we immediately see that it is impossible
to obtain Bell’s inequality without additional assumptions; see also [27].

We remark that generalized Bell-type inequalities can be obtained that are not violated by experimental
statistical data taken from different experiments [21–23].

In this paper I would like to continue analyzing Bell’s arguments. I would like to formalize the rules
of classical → quantum correspondence. And we will see that it is not so simple a task. The first step in
this direction was taken by von Neumann when he formulated the first NO-GO statement for existence
of a prequantum classical statistical model [30]. J. Bell continued this activity by starting with harsh
critical arguments against von Neumann’s formalism. We proceed in the same way. Our analysis showed
that Bell’s formalism was far from complete.

Since the aim of this paper is to attract the attention of physicists to the probabilistic structure of