STAR-PRODUCT; SYMPLECTIC AND SPIN TOMOGRAPHIES∗

Olga V. Man’ko

P. N. Lebedev Physical Institute, Russian Academy of Sciences
Leninskii Prospect 53, Moscow 119991, Russia
e-mail: omanko@sci.lebedev.ru

Abstract

Tomographic probabilities for both discrete spin variables and continuous quadratures are considered within the framework of the star-product quantization.

Keywords: tomograms, spin, photon quadratures, star-product, symplectic tomography, spin tomography.

1. Introduction

The star-product quantization of classical systems (see, e.g., [1–5]) provides the possibility to use functions on the phase space (instead of operators) to describe physical observables in quantum mechanics and quantum field theory. An example of such approach is employing the Wigner quasidistribution function [6] instead of the density operator for describing quantum states. Recently it was discovered [7,8] that the star-product scheme can be used to replace the description of quantum states, in view of the Wigner quasidistribution function, by the standard positive measurable probability distribution which is called symplectic tomogram [9] of quantum state, e.g., of the photon or the harmonic oscillator. In [10–12] the analogous description of quantum spin states by probability distributions was suggested. The scheme of spin tomography is also an example of the particular star-product approach.

Our aim here is to present a review of the symplectic and spin tomographic approaches within the framework of star-product quantization following [7, 8].

This paper is organized as follows.

In Sec. 2 the generic star-product scheme is reviewed. In Sec. 3 symplectic tomography of systems with continuous variables like the oscillator coordinates and photon quadratures is presented. In Sec. 4 spin tomography is considered. The conclusions and perspectives are given in Sec. 5.

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2. Star-Product Scheme

Let us construct the bijective map

\[ \hat{A} \longleftrightarrow f_{\hat{A}}(x), \]  

(1)

where \( \hat{A} \) is an operator acting in a Hilbert space \( \mathcal{H} \) and \( f_{\hat{A}}(x) \) is the function of vector variables \( x = (x_1, x_2, \ldots, x_n) \). This function is called the symbol of the operator \( \hat{A} \). To construct such a map, one can use two sets of operators \( \hat{U}(x) \) and \( \hat{D}(x) \) acting in the Hilbert space \( \mathcal{H} \). The operators \( \hat{U}(x) \) are called dequantizers since they map the operator \( \hat{A} \) onto the operator symbol

\[ f_{\hat{A}}(x) = \text{Tr} \left[ \hat{A} \hat{U}(x) \right]. \]  

(2)

The operators \( \hat{D}(x) \) are called quantizers since they map the function \( f_{\hat{A}}(x) \) onto the operator, namely,

\[ \hat{A} = \int f_{\hat{A}}(x) \hat{D}(x) \, dx. \]  

(3)

Formulas (2) and (3) are self-consistent if one has the following property of the quantizers and dequantizers:

\[ \text{Tr} \left[ \hat{U}(x) \hat{D}(x') \right] = \delta(x - x'). \]  

(4)

The Dirac delta-function in (4) is used for the case of continuous variables \( x \). In the case of discrete variables \( x \), one uses the Kronecker symbol \( \delta_{xx'} \) in (4) and the sum instead of integral in (3). The map (2), (3) provides the following nonlocal product of two operator symbols:

\[ f_{\hat{A}}(x) * f_{\hat{B}}(x) = \int f_{\hat{A}}(x_1) f_{\hat{B}}(x_2) K(x_1, x_2, x) \, dx_1 \, dx_2. \]  

(5)

Here the kernel of the nonlocal product (5) is connected with the quantizers and dequantizers by the formula

\[ K(x_1, x_2, x) = \text{Tr} \left[ \hat{D}(x_1) \hat{D}(x_2) \hat{U}(x) \right]. \]  

(6)

The product (5) is called the star-product of the functions. There is a well-known example of the so-called Weyl–Wigner–Moyal star-product of functions \( f_{\hat{A}}(q, p) \) and \( f_{\hat{B}}(q, p) \), for which \( x = (q, p) \) and the dequantizer \( \hat{U}(q, p) \) reads

\[ \hat{U}(q, p) = 2 \exp \left( 2(\alpha \hat{a}^+ - \alpha^* \hat{a}) \right) \hat{I}. \]  

(7)

Here \( \hat{I} \) is the parity operator and \( \hat{a} \) and \( \hat{a}^+ \) are the annihilation and creation operators for the harmonic oscillator, i.e.,

\[ \hat{a} = \frac{1}{\sqrt{2}} (\hat{q} + i\hat{p}), \quad \hat{a}^+ = \frac{1}{\sqrt{2}} (\hat{q} - i\hat{p}), \]  

(8)

where \( \hat{q} \) and \( \hat{p} \) are dimensionless position and momentum operators. The complex numbers \( \alpha \) and \( \alpha^* \) are expressed in terms of the phase-space coordinates

\[ \alpha = \frac{1}{\sqrt{2}} (q + ip), \quad \alpha^* = \frac{1}{\sqrt{2}} (q - ip). \]  

(9)

The quantizer \( \hat{D}(q, p) \) is expressed in terms of the dequantizer \( \hat{U}(q, p) \) as follows:

\[ \hat{D}(q, p) = \frac{\hat{U}(q, p)}{2\pi}. \]  

(10)