The Cavity Method for the Rigidity Transition
J. Barre,1 A. R. Bishop,1 T. Lookman1, and A. Saxena1

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Abstract: We have used the cavity method to study the floppy to rigid transition in a 2-dimensional (2D) random graph as well as in a 3D small world chain. Our analytic results are in excellent agreement with numerical studies using the pebble game algorithm. We also illustrate that a transfer matrix method is equivalent to the cavity method at the replica symmetric level.

KEY WORDS: Rigidity transition; combinatorial optimization; cavity method.

1. INTRODUCTION

The rigidity problem dates back at least to the 19th century, when Maxwell made a decisive contribution. The problem can be presented as follows. Consider an ensemble of atoms, which can a priori move freely in the plane, or in 3D space; these atoms are linked by a certain number of rigid bonds. One would like to know if the resulting network is rigid or floppy, that is, can it bear an applied stress or not? It is clear that when there are too few bonds, the network is floppy, and will continuously deform without energy cost under stress; as bonds are added, it becomes rigid: this is the rigidity transition. Rigidity theory has been very successfully applied to the study of network glasses, and has also recently been used to analyze proteins.

Although the rigidity problem is physically clear, analytic approaches are difficult and few. Maxwell introduced a mean-field like approach, which consists of counting the number of degrees of freedom of the system and the number of constraints induced by the bonds. The transition is
Critical exponents were estimated by renormalization group analysis. (5) To our knowledge, exact solutions have been obtained only for Random Bond Models, which have no finite size loops in the thermodynamic limit, thus allowing for the use of Cayley tree techniques. Duxbury et al. and Chubinsky et al. solved several types of Random Bond Models using a transfer matrix method. (6–8)

The rigidity problem bears striking similarities with a number of combinatorial problems. We briefly introduce one of them, K-SAT. An instance of the K-SAT problem involves $N$ Boolean variables $x_i \in \{0, 1\}$, and $M$ constraints. The constraints take the form of "OR" functions of $K$ variables, for instance $x_1 \lor \neg x_3 \lor x_9$, if $K = 3$ ($\neg x_3$ means NOT $x_3$).

The instance is said to be satisfiable if there exists an assignment of the variables $x_i$ such that all the constraints are satisfied, and unsatisfiable otherwise. The problem is to know whether a given instance is satisfiable or not, and the answer obviously depends on the ratio $M/N$ (constraints/variables). The analogy with the rigidity problem is easy to draw: Boolean variables correspond to the atoms and their degrees of freedom; logical constraints to the physical constraints imposed by the bonds; a satisfiable instance of K-SAT to a floppy network; an unsatisfiable instance to a rigid one; and finally the control parameter $M/N$ to the average number of bonds per atom. Recently, important analytical breakthroughs have been made for K-SAT and other combinatorial problems using a method borrowed from spin glass theory, termed the cavity method. (9)

It is then interesting to ask if the cavity method can help to solve rigidity models. It turns out that this cavity method is essentially equivalent to the transfer matrix one developed in refs. 6–8 for the rigidity problem.

The similarity between K-SAT and the rigidity problem has been recognized in the literature, as cross-citations from articles in both fields testify. (10, 11) It seems, however, that the analogy has not been developed in depth: the goal of this paper is to provide a firmer foundation, by demonstrating clearly the link between the transfer matrix method and the cavity method. This will bring the analogy to an operational stage, where ideas and methods can be transferred from one field to the other. In particular, the concept of an "intermediate phase" as a precursor to the transition recently emerged in both fields; (9, 12) we will discuss the possible similarity between them. Also, different conjectures have been made for combinatorial problems concerning the link between a phase transition and the onset of computational complexity. We will assess where the rigidity problem stands in this respect. Finally, we present a solution for the rigidity transition of a small world network using the cavity method.