Asymptotic Behavior of the Order Parameter in a Stochastic Sandpile

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We derive the first four terms in a series for the order parameter \( \rho \), the stationary activity density in the supercritical regime of a one-dimensional stochastic sandpile; in the two-dimensional case the first three terms are reported. This is done by reorganizing the perturbation theory derived using a path-integral formalism \cite{Dickman and Vidigal, J. Phys. A 35, 7269 (2002)}, to obtain an expansion for stationary properties. Since the process has a strictly conserved particle density \( p \), the Fourier mode \( N^{-1} \psi_k = 0 \rightarrow p \), when \( N \rightarrow \infty \), and so is not a random variable. Isolating this mode, we obtain a new effective action leading to an expansion for \( \rho \) in the parameter \( \kappa \equiv 1/(1 + 4p) \). This requires enumeration and numerical evaluation of more than 200,000 diagrams, for which task we develop a computational algorithm. Predictions derived from this series are in good accord with simulation results. We also discuss the nature of correlation functions and one-site reduced distributions in the small-\( \kappa \) (high-density) limit.

KEY WORDS: Sandpiles; series expansion; path integrals; stochastic processes; phase transitions.

1. INTRODUCTION

Sandpile models were introduced some 15 years ago as examples of self-organized criticality (SOC), or scale invariance in the apparent absence of adjustable parameters.\cite{1–5} Although not directly related to real sand or other granular systems, these models have attracted great interest in the quest for understanding the ubiquity of power-law distributions in nature\cite{6}, for example in earthquakes.\cite{7,8}

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Subsequently the appearance of "spontaneous" criticality in sandpiles was shown to result from a control mechanism that forces the system to a critical point marking a phase transition to an absorbing state. This absorbing-state phase transition is observed in sandpiles with the same local dynamics as the original self-organized versions, but with a strictly conserved particle density, \( p \), which plays the role of a temperature-like control parameter. Sandpiles with a strictly conserved particle number (to be called "conserved sandpiles" in what follows) are not self-organized: to achieve criticality, the particle density must be adjusted to its critical value, just as the temperature must be adjusted in a fluid or magnetic system. The particle addition and loss rules in self-organized sandpiles amount to a control scheme that forces \( p \) to its critical value.\(^{(9)}\)

Conserved sandpiles, as noted, exhibit an absorbing-state phase transition,\(^{(12–14)}\) analogous to that of the contact process or directed percolation (DP). Although the DP universality class is generic for absorbing-state phase transitions,\(^{(15,16)}\) simulation results suggest that conserved sandpiles belong to universality classes other than DP. The difference is commonly attributed to the presence of a conserved field (the particle density), but there is as yet no firm basis for this assertion. Understanding criticality in conserved sandpiles thus presents an interesting challenge to the ongoing program of understanding nonequilibrium universality classes.\(^{(12,13,17)}\)

For the stochastic sandpile to be studied here, simulation results yield larger values for the critical exponent \( \beta \) associated with the order parameter, (about 0.39 and 0.64 in one and two dimensions, respectively), than the corresponding DP values (0.2765 and 0.583 in one and two dimensions). DP-like critical behavior has been established for sandpiles with "sticky grains" (in this case above-threshold sites do not always topple).\(^{(18)}\)

Finally, the Bak–Tang–Wiesenfeld model, with a deterministic toppling rule, appears to define its own universality class.\(^{(19,20)}\) In the case of conserved sandpiles, a systematic renormalization group analysis using the epsilon expansion is as yet unavailable. In fact, even the value of the upper critical dimension remains controversial.\(^{(11,21–23)}\) Conflicting critical exponent values have been reported for the one-dimensional stochastic conserved sandpile,\(^{(24–27)}\) possibly reflecting finite-size effects. Until now, most quantitative results for conserved sandpiles have been based on simulations,\(^{(19,24–26,28,29)}\) an important exception being the solution by Priezzhev et al.\(^{(30)}\) of a directed, conserved version of the Maslov–Zhang model\(^{(31)}\) via the Bethe ansatz. Phenomenological field theories have been proposed for sandpiles,\(^{(11,21,22)}\) but their analysis is far from straightforward. Given the conflicting simulation results in the literature regarding critical exponents for conserved stochastic sandpiles, it is of interest to develop alternative approaches.