Metastability for Reversible Probabilistic Cellular Automata with Self-Interaction

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Abstract The problem of metastability for a stochastic dynamics with a parallel updating rule is addressed in the Freidlin–Wentzel regime, namely, finite volume, small magnetic field, and small temperature. The model is characterized by the existence of many fixed points and cyclic pairs of the zero temperature dynamics, in which the system can be trapped in its way to the stable phase. Our strategy is based on recent powerful approaches, not needing a complete description of the fixed points of the dynamics, but relying on few model dependent results. We compute the exit time, in the sense of logarithmic equivalence, and characterize the critical droplet that is necessarily visited by the system during its excursion from the metastable to the stable state. We need to supply two model dependent inputs: (1) the communication energy, that is the minimal energy barrier that the system must overcome to reach the stable state starting from the metastable one; (2) a recurrence property stating that for any configuration different from the metastable state there exists a path, starting from such a configuration and reaching a lower energy state, such that its maximal energy is lower than the communication energy.

Keywords Stochastic dynamics · Probabilistic cellular automata · Metastability · Low temperature dynamics

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1 Introduction

Metastable states are very common in nature and are typical of systems close to a first order phase transition. It is often observed that a system can persist for a long period of time in a phase which is not the one favored by the thermodynamic parameters; classical examples are the super-saturated vapor and the magnetic hysteresis. The rigorous description of this phenomenon in the framework of well defined mathematical models is relatively recent, dating back to the pioneering paper \cite{3}, and has experienced substantial progress in the last decade. See \cite{12} for a list of the most important papers on this subject.

A natural setup in which the phenomenon of metastability can be studied is that of Markov chains, or Markov processes, describing the time evolution of a statistical mechanical system. Think for instance to a stochastic lattice spin system. In this context powerful theories (see \cite{2,9,11}) have been developed with the aim to find answers valid with maximal generality and to reduce to a minimum the number of model dependent inputs necessary to describe the metastable behavior of the system. Whatever approach is chosen, the key model dependent question is the computation of the minimal energy barrier, called communication energy, to be overcome by a path connecting the metastable to the stable state. Such a problem is in general quite complicated and becomes particularly difficult when the dynamics has a parallel character. Indeed, if simultaneous updates are allowed on the lattice, then no constraint on the structure of the trajectories in the configuration space is imposed. Therefore, to compute the communication energy, one must take into account all the possible transitions in the configuration space.

The problem of the computation of the communication energy in a parallel dynamics setup has been addressed in \cite{4,5}. In particular, in \cite{5} the typical questions of metastability, that is the determination of the exit time and of the exit tube, have been answered for a reversible Probabilistic Cellular Automaton (see \cite{6,8,10,13–15}), in which each spin is coupled only with its nearest neighbors. In that paper it has been shown that, during the transition from the metastable minus state to the stable plus state, the system visits an intermediate chessboard-like phase. In the present paper we study the reversible PCA in which each spin interacts both with itself and with its nearest neighbors; the metastable behavior of such a model has been investigated on heuristic and numerical grounds in \cite{1}. The addition of the self-interaction changes completely the metastability scenario; in particular we show that the chessboard-like phase plays no role in the exit from the metastable phase.

Another very interesting feature of this model is the presence of a large number of fixed points of the zero-temperature dynamics in which the system can be trapped. Following the powerful approach of \cite{9}, we can compute the exit time avoiding a complete description of the trapping states. However, we cannot describe the exit tube, i.e., the tube of trajectories followed by the system during its exit from the metastable to the stable phase. The only information on the exit path that we prove in this paper is the existence of a particular set of configurations which is necessarily visited by the system during its excursion from the metastable to the stable state. This set plays the role of the saddle configuration set, which is usually introduced in the study of the metastable behavior of sequential dynamics.

According to the approach of \cite{9}, the model dependent ingredients that must be provided are essentially two: (1) the solution of the global variational problem for all the paths connecting the metastable and the stable state, i.e., the computation of the communication energy; (2) a sort of recurrence property stating that, starting from each configuration different from the metastable and the stable state, it is possible to reach a configuration at lower energy following a path with an energy cost strictly smaller than the communication energy.

To solve the global variational problem (see items 2 and 3 in Theorem 2.3), we obtain an upper bound on the communication energy by exhibiting a path connecting the metastable...