Coercivity Conditions for Equilibrium Problems

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Abstract. The study of the existence of solutions of equilibrium problems on unbounded domains involves usually the same sufficient assumptions as for bounded domains together with a coercivity condition. We focus on two different conditions: the first is obtained assuming the existence of a bounded set such that no elements outside is a candidate for a solution; the second allows the solution set to be unbounded. Our results exploit the generalized monotonicity properties of the function $f$ defining the equilibrium problem. It turns out that, in both the pseudomonotone and the quasimonotone setting, an equivalence can be stated between the nonemptyness and boundedness of the solution set and these coercivity conditions. In the pseudomonotone case, we compare our coercivity conditions with various coercivity conditions that appeared in the literature.

Key Words. Equilibrium problems, generalized monotonicity, generalized convexity, coercivity conditions.

1. Introduction

Let $X$ be a reflexive Banach space, let $K$ be a closed, convex subset of $X$, and let $f : K \times K \to \mathbb{R}$. By equilibrium problem, we understand the problem of finding $\bar{x} \in K$ such that

\[(EP) \quad f(\bar{x}, y) \geq 0, \quad \forall y \in K.\]

It is well known that the EP is related closely to the following problem, termed the dual equilibrium problem: find $\bar{x} \in K$ such that

\[(DEP) \quad f(y, \bar{x}) \leq 0, \quad \forall y \in K.\]
We denote by $S_K$ the solutions of EP and by $S^D_K$ the solutions of DEP.

Classical examples of equilibrium problems are variational inequalities, optimization problems, complementarity problems. In particular, in the case of variational inequalities,

$$f(x, y) = \langle A(x), y - x \rangle, \quad \text{with } A: X \rightarrow X^*.$$  

The existence of solutions for equilibrium problems on unbounded domains involves usually the same conditions used for bounded domains, together with a coercivity condition. Classical results about the existence of solutions for variational inequalities on noncompact subsets of $\mathbb{R}^n$ are described in the survey by Harker and Pang (Ref. 1).

Existence of solutions for pseudomonotone variational inequalities was established first by Cottle and Yao (Ref. 2). More recently, results aimed at finding sharp coercivity conditions for pseudomonotone variational inequalities can be found in Ref. 3 for reflexive Banach spaces and in Ref. 4 for finite-dimensional spaces; in Ref. 4, in particular, a coercivity condition involving the recession cone of the feasible set is considered. In Ref. 5, the case of quasimonotone variational inequalities is studied; exploiting recent results about the existence of solutions that appeared in Ref. 6, sharp coercivity conditions are provided. In the case of generalized equilibrium problems, Flores-Bazán (Ref. 7) provided a characterization of the nonemptiness of the solutions set by using an approach based on recession notions coming from minimization problems.

As in Refs. 3 and 5, the purpose of this paper is to find coercivity conditions as weak as possible, exploiting the generalized monotonicity properties of the function $f$ defining the equilibrium problem. In Section 2, we present some results needed in the sequel. Section 3 deals with the case of pseudomonotone functions, while Section 4 is devoted to the quasimonotone case; in these sections, characterizations of the nonemptiness and boundedness of the set of equilibria are given in terms of suitable coercivity conditions. Finally, in Section 5, the coercivity conditions that appeared in literature in the context of variational inequalities or optimization problems, and adapted to equilibrium problems, are compared with ours.

2. Preliminary Results

Applying to equilibrium problems a recent definition introduced in Ref. 6 in the case of variational inequalities, we denote by $S^D_{K, \text{loc}}$ the local solutions of the dual problem; i.e.,