Sufficient Optimality Criterion for Linearly Constrained, Separable Concave Minimization Problems

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Abstract. A sufficient optimality criterion for linearly-constrained concave minimization problems is given in this paper. Our optimality criterion is based on the sensitivity analysis of the relaxed linear programming problem. The main result is similar to that of Phillips and Rosen (Ref. 1); however, our proofs are simpler and constructive.

In the Phillips and Rosen paper (Ref. 1), they derived a sufficient optimality criterion for a slightly different linearly-constrained concave minimization problem using exponentially many linear programming problems. We introduce special test points and, using these for several cases, we are able to show optimality of the current basic solution.

The sufficient optimality criterion described in this paper can be used as a stopping criterion for branch-and-bound algorithms developed for linearly-constrained concave minimization problems.

Key Words. Separable concave minimization problems, linear relaxation, sensitivity analysis.

1. Introduction

We consider separable concave minimization problems in the following form:

\[ \min \{ f(x) : \mathbf{Ax} = \mathbf{b}, \ x \in \mathbb{R}^n \} \]

where \( f(x) = \sum_{i=1}^{m} f_i(x_i) \) is a separable concave function, \( \mathbf{A} \) is an \( m \times n \) matrix, \( \mathbf{b} \) is an \( m \times 1 \) vector, and \( x \) is an \( n \times 1 \) vector.

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\[ \min \sum_{j=1}^{n} f_j(x_j), \]
\[ \text{s.t. } Ax \leq b, \]
\[ l \leq x \leq u, \]
where \( A \in \mathbb{R}^{m \times n} \) is a matrix, \( b \in \mathbb{R}^m \), \( l, u \in \mathbb{R}^n \) are given vectors, \( l \geq 0 \), \( f_j : \mathbb{R} \to \mathbb{R} \) are concave functions, and \( x \in \mathbb{R}^n \) is a vector of unknowns. Let us introduce the sets
\[ A = \{ x \in \mathbb{R}^n : Ax \leq b \}, \]
\[ T = \{ x \in \mathbb{R}^n : l \leq x \leq u \} . \]
Then, the set of feasible solutions of problem (P) is defined as
\[ P = A \cap T , \]
which assumes that the domain of \( f_j \) \( \subseteq D_{f_j} \) holds. Furthermore, (P) assumes the set of optimal solutions of problem (P) \( P^* \neq \emptyset \) does not belong to the class of convex optimization. This problem has two important theoretical properties: there is an optimal solution at a vertex of the polyhedron \( P \); moreover, if \( f_j \) is strictly concave, then each optimal solution is a vertex of the polyhedron. Hence, problem (P) is in the class of NP-complete problems. Several practical applications can be formulated via problem (P), for instance; certain control problems, concave knapsack problems, various production and transportation problems, production planning problems, process network synthesis problems, some network flow problems. Due to the importance and applicability of model (P), there is a considerable literature on possible solution methods. In the literature, there are three main types of algorithm: listing the vertices of the polyhedron \( P \), cutting-plane methods, and branch-and-bound algorithms (BB). Several versions of BB are discussed in Refs. 1 and 10–16; vertex enumeration procedures are used in Refs. 17–19 to solve problem (P); cutting-plane algorithms are described in Refs. 20–22. There are a number of alternative methods such as approximation using splines (Ref. 23) or combination of BB and cutting-plane algorithms (Ref. 21).

In this paper, a sufficient optimality criterion is given for linearly-constrained separable concave minimization problem. The optimality criterion is based on the linear programming relaxation of (P) and the sensitivity analysis of that linear programming problem. The result obtained is similar to that of Phillips and Rosen (Ref. 1), but our proof is elementary and