Modified Dinkelbach-Type Algorithm for Generalized Fractional Programs with Infinitely Many Ratios

J. Y. Lin

and R. L. Sheu

Communicated by S. Schaible

Abstract. In this paper, we extend the Dinkelbach-type algorithm of Crouzeix, Ferland, and Schaible to solve minmax fractional programs with infinitely many ratios. Parallel to the case with finitely many ratios, the task is to solve a sequence of continuous minmax problems,

\[ P(\alpha^k) = \min_{x \in X} \left( \max_{t \in T} \left[ f_t(x) - \alpha^k g_t(x) \right] \right) , \]

until \{\alpha^k\} converges to the root of \( P(\alpha) = 0 \). The solution of \( P(\alpha^k) \) is used to generate \( \alpha^{k+1} \). However, calculating the exact optimal solution of \( P(\alpha^k) \) requires an extraordinary amount of work. To improve, we apply an entropic regularization method which allows us to solve each problem \( P(\alpha^k) \) incompletely, generating an approximate sequence \{\tilde{\alpha}^k\}, while retaining the linear convergence rate under mild assumptions. We present also numerical test results on the algorithm which indicate that the new algorithm is robust and promising.

Key Words. Generalized fractional programming, minmax problems, entropic regularization.

1. Introduction

A generalized fractional program, also known as minmax fractional program, is of the following form:

\[ (P) \min_{x \in X} \max_{t \in T} \left\{ \frac{f_t(x)}{g_t(x)} \right\} , \]

1 This research was partially supported by the National Science Council of Taiwan under Project NSC 91-2215-M-006-017.

2 PhD Candidate, Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan.

3 Professor, Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan.
where $X$ is a compact subset in $\mathbb{R}^n$, $T$ is a compact metric space, and the functions $f_t(x), g_t(x)$ are continuous on $T \times X$ with $g_t(x) > 0$ on $X$ for all $t \in T$. Notice that $T$ can be a finite set, which is obviously a compact metric space in the discrete topology and thus can be included in our discussion of model (P).

In the literature, most results were derived for the case of a finite set $T$. The pioneering work was due to Crouzeix, Ferland, and Schaible (Refs. 1–2). A survey of generalized fractional programming can be found in Refs. 3–5. For more recent developments, including cutting planes, dual, or other approaches, see Refs. 6–10.

Up to now, we have not seen any algorithm for solving the fractional program (P) with a continuous index set $T$. Our work extends the classical Dinkelbach-type algorithm of Refs. 1–2 to a more general $T$; but this cannot be taken too far without proper assumptions. We have a pathological example, short of the compactness assumption for $T$, which ends up with the root of $P(\alpha) = \inf_{x \in X} \left( \sup_{t \in T} \left[ f_t(x) - \alpha g_t(x) \right] \right) = 0$ failing to be the optimal value of (P). This phenomenon contradicts the most fundamental property of a Dinkelbach-type algorithm. Fortunately, this is the worst which can happen. The above pathological example can be excluded under the compactness assumption. In fact, even for a finite $T$, compactness was used. It is only too trivial to notice in the discrete topology.

In practice, the implementation of the algorithm becomes far more complex for a continuous $T$ than in the discrete case. The iterative nature of the Dinkelbach-type algorithm worsens the situation considerably. Here, one must first solve $(P(\alpha_k)) = \min_{x \in X} \left( \max_{t \in T} \left[ f_t(x) - \alpha_k g_t(x) \right] \right)$ with respect to a sequence of parameters $\{\alpha_k\}$. The solution $x_k$ is then used to determine the next parameter as $\alpha_{k+1} = \max_{t \in T} \{f_t(x_k)/g_t(x_k)\}$.

Such a procedure may lose its edge, should the cost of generating $x_k$ be too expensive to justify itself. This is very likely so when $T$ is continuous.

To resolve this issue, our idea is to solve (1) only approximately at each step. The difficulty of designing a relaxation method lies in assuring its convergence. For example, if an interim solution $\tilde{x}_k$ has a negative objective value, i.e.,