Zero-Sum Differential Games Involving Hybrid Controls

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Abstract. We study a zero-sum differential game with hybrid controls in which both players are allowed to use continuous as well as discrete controls. Discrete controls act on the system at a given set of interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, an autonomous jump set $A$ or a controlled jump set $C$, where one controller can choose to jump or not. At each jump, the trajectory can move to a different Euclidean space. One player uses all the three types of controls, namely, continuous controls, autonomous jumps, and controlled jumps; the other player uses continuous controls and autonomous jumps. We prove the continuity of the associated lower and upper value functions $V^-$ and $V^+$. Using the dynamic programming principle satisfied by $V^-$ and $V^+$, we derive lower and upper quasivariational inequalities satisfied in the viscosity sense. We characterize the lower and upper value functions as the unique viscosity solutions of the corresponding quasivariational inequalities. Lastly, we state an Isaacs like condition for the game to have a value.

Key Words. Dynamic programming principle, viscosity solutions, quasivariational inequalities, hybrid control, differential games.

1 This work was partially supported by Grants DRDO 508 and ISRO 050 to the Nonlinear Studies Group, Indian Institute of Science. The first author is a University Grant Commission Research Fellow and the financial support is gratefully acknowledged.

2 The authors thank Prof. M.K. Ghosh, Department of Mathematics, Indian Institute of Science, for introducing the problem and thank the referee for useful suggestions.

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1. Introduction
We study here a zero sum differential game involving hybrid controls. The main motivation comes from the hybrid control systems arising in many engineering problems like constrained robotic systems, automated highway systems, and flight control systems; see Ref. 1. Branicky, Borkar, and Mitter (Ref. 1) have presented a model of a hybrid control system where continuous controls and discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, namely, an autonomous jump set $A$ or a controlled jump set $C$, where the controller can choose to jump or not; at each jump, the trajectory can move to a different Euclidean space. The problem of Ref. 1 is to find the value function of a system described by differential equations, jump sets, and a cost function. The authors of Ref. 1 prove the right continuity of the value function. By using the dynamic programming principle, they have arrived at a quasivariational inequalities satisfied by the value functions in the viscosity sense. In Ref. 2, Bensoussan and Menaldi study a similar system and prove that the value function is continuous. They prove the uniqueness for a special case when the autonomous jump set is empty. For this hybrid control problem, we have shown in Ref. 3 that the value function is Hölder continuous under a transversality condition. We have proved the uniqueness of the value function by a method very different from that in Ref. 2 and for a more general case, namely the autonomous jump set being nonempty.

In this work, we extend the model in Ref. 1 to the game theoretic set up. We allow player $P_1$ to use all the three types of controls, namely continuous controls, autonomous jumps, and controlled jumps; we allow the other player $P_2$ to use continuous controls and autonomous jumps. As in Ref. 4, we use the Elliot-Kalton approach to define the strategies and the lower and upper value functions. The viscosity solution techniques applied to differential game problems using the Elliot-Kalton strategies and the dynamic programming principle give good existence and uniqueness results for the Hamilton-Jacobi-Isaacs equations satisfied by the value functions of the game problem; see chapter 6 of Ref. 4 and the references therein. Games where both players use continuous controls (Refs. 4–5), when both players use switching strategies (Ref. 6), and a problem where one player uses impulse controls and the other use continuous control and/or switching controls (Ref. 7) are particular examples, to name a few.

In this paper, our aim is to prove the local Hölder continuity of the lower and upper value functions, to derive the corresponding lower and upper quasivariational inequalities (QVIs) satisfied by them in the viscosity sense, and to characterize them as the unique viscosity solutions of these QVIs. Finally, we give an Isaacs like condition for the upper and lower value to coincide and thus for the game to have a value.