Necessary Optimality Conditions in Terms of Convexificators in Lipschitz Optimization

X. F. Li and J. Z. Zhang

Abstract. This study is devoted to constraint qualifications and Kuhn-Tucker type necessary optimality conditions for nonsmooth optimization problems involving locally Lipschitz functions. The main tool of the study is the concept of convexificators. First, the case of a minimization problem in the presence of an arbitrary set constraint is considered by using the contingent cone and the adjacent cone to the constraint set. Then, in the case of a minimization problem with inequality constraints, Abadie type constraint qualifications and several other qualifications are proposed; Kuhn-Tucker type necessary optimality conditions are derived under the qualifications.

Key Words. Locally Lipschitz functions, Dini derivatives, convexificators, constraint qualifications, necessary optimality conditions, nonsmooth optimizations.

1. Introduction

Recently, the idea of convexificators has been employed to extend and strengthen various results in nonsmooth analysis and optimization (see Refs. 1–13). The notion of convexificators was introduced in Ref. 1, further developed in Ref. 7, and extended in Ref. 6, where a new concept of approximate Jacobians is introduced that is a generalization of the concept of convexificators of real-valued functions to vector-valued maps. Convexificators, generating upper convex and lower concave approximations to a function at a point, can be viewed as a means to extend and strengthen various results in nonsmooth analysis and optimization.
as weaker versions of the notion of subdifferentials that so much pervades the study of nonsmooth analysis. It has been shown in Ref. 7 (see also Refs. 3 and 12) that the Clarke subdifferential (Ref. 14), Michel-Penot subdifferential (Ref. 15), Ioffe-Morduchovich subdifferential (Refs. 16 and 17), and Treiman subdifferential (Ref. 18) of a locally Lipschitz real-valued function are convexificators and that the Clarke generalized Jacobian (Ref. 14) and $B$-subdifferential (Ref. 19) of a locally Lipschitz vector-valued map are approximate Jacobians. It has been shown also in these papers (Refs. 3, 7, 12) that these known subdifferentials of locally Lipschitz functions and generalized Jacobians of locally Lipschitz maps may contain often respectively the convex hull of a convexificator and that of an approximate Jacobian. Therefore, from the point of view of optimization and its applications, the descriptions of the optimality conditions and calculus rules in terms of convexificators and approximate Jacobians provide sharp results. For nonsmooth optimization problems, various results concerning Fritz-John type and Kuhn-Tucker type necessary optimality conditions that use convexificators or approximate Jacobians have been developed in Refs. 6, 8, 10–13.

The present paper is devoted to constraint qualifications and Kuhn-Tucker type necessary optimality conditions for minimization and maximization problems with arbitrary set constraints or inequality constraints, where the objective and the constraint functions are real-valued and locally Lipschitz continuous. The qualifications and the necessary optimality conditions are stated in terms of upper or lower convexificators.

In Section 3 of this paper, for a minimization problem with an arbitrary (not necessarily convex) set constraint, Kuhn-Tucker type necessary optimality conditions are developed that use upper convexificators and closed convex subcones of the contingent cone or those of the adjacent cone to the constraint set. An example is given to illustrate that generally the closed convex subcones in the necessary optimality conditions cannot be replaced by the contingent cone or the adjacent cone to the constraint set. For arbitrary set-constrained minimization problems involving locally Lipschitz objective functions, Kuhn-Tucker type necessary optimality conditions have been developed in Ref. 12 by using upper convexificators, and in Ref. 20 by using the Clarke subdifferentials and closed convex subcones of the contingent cone to the constraint set. The results of the necessary optimality conditions developed in this section are different from the corresponding one in Ref. 12 and extend the corresponding one in Ref. 20.

In Section 4, the results obtained in Section 3 are applied to develop Kuhn-Tucker type necessary optimality conditions for a minimization problem with only inequality constraints. In this section, Abadie type qualifications, Cottle type qualification, Slater type qualification, linearity type qualification, and linear independence type qualification are proposed; interrelationships of these qualifications are also presented. Since these qualifications are given by using upper convexificators and/or generalized convexity of functions introduced in this paper, they