TECHNICAL NOTE

Characterization of the Cone of Attainable Directions\(^1\)

B. Jiménez\(^2\) and V. Novo\(^3\)

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Abstract. A useful characterization of the cone of attainable directions is provided.

Key Words. Cone of attainable directions, Ursescu cone, intermediate cone, adjacent cone, tangent cone, inner tangent cone.

1. Introduction

The notion of tangent cone to a set \(S\) at a point \(x_0\) plays a crucial role in optimization. One of the most used cones is the cone of attainable directions. A modified version of this cone was introduced by Kuhn and Tucker in 1951 in order to establish the necessary optimality conditions. These authors considered a differentiable arc contained in \(S\). Subsequently, this definition has been weakened considering a continuous function on an interval and differentiable at 0 or verifying only the last property (which implies the continuity at the point 0, but not on a neighborhood). Since the 1930s, the tangent cone or Bouligand cone has been a usual tool in optimization; for this cone, a sequence in \(S\) converging to \(x_0\) in a certain direction (instead of a differentiable function) is considered (see Definition 2.1).

In this paper, we provide a characterization of the cone of attainable directions. This characterization breaks the traditional structure of quantificators,
converting “for all \( \{ t_n \} \) there exists \( \{ v_n \} \)” into “there exists \( \{ t_n \} \in C \) and there exists \( \{ v_n \} \),” for a suitable class \( C \) of sequences \( \{ t_n \} \).

2. Main Result

Let \( E \) be a normed linear real space, let \( a \in \mathbb{R} \), and let \( \varphi \) be a function from \( [a, a + \delta) \) to \( E \) for some \( \delta > 0 \). We will use the following derivatives of \( \varphi \) at the point \( a \):

\[
\varphi'(a) = \lim_{t \downarrow 0} \frac{\varphi(a + t) - \varphi(a)}{t},
\]

\[
\varphi'_+(a) = \limsup_{t \downarrow 0} \frac{\varphi(a + t) - \varphi(a)}{t},
\]

\[
\varphi'_-(a) = \liminf_{t \downarrow 0} \frac{\varphi(a + t) - \varphi(a)}{t}.
\]

This paper is focused on the following notion.

Definition 2.1. Let \( S \subset E \) and let \( x_0 \in \text{cl} \ S \) (\( \text{cl} \ S \) designates the closure of \( S \)).

(a) The tangent cone to \( S \) at \( x_0 \) is

\[
T(S, x_0) = \{ v \in E : \exists \{ t_n \} \downarrow 0, \exists \{ v_n \} \to v \text{ such that } x_0 + t_n v_n \in S, \forall n \in \mathbb{N} \}.
\]

(b) The cone of attainable directions to \( S \) at \( x_0 \) is

\[
A(S, x_0) = \{ v \in E : \exists \delta > 0, \exists \varphi : [0, \delta) \to E \text{ such that } \varphi(0) = x_0, \varphi(t) \in S, \forall t \in (0, \delta), \varphi'(0) = v \}.
\]

In the literature, the previous cones have received different names. Thus, the tangent cone is also called the contingent cone, the Bouligand tangent cone, the outer tangent cone, and the cone of adherent displacements; the cone of attainable directions is also called the adjacent cone, the intermediate cone, the inner tangent cone, and the Ursescu cone.

Let us observe that the only regularity requirement on the function \( \varphi \) in Definition 2.1 (b) is

\[
\lim_{t \downarrow 0} \frac{\varphi(t) - \varphi(0)}{t} = v,
\]

i.e., the existence of the right-hand derivative at 0. This condition implies that \( \varphi \) is continuous at 0, but nothing is said about the continuity or differentiability of \( \varphi \) on \( (0, \delta) \).