Positive Principal Minor Property of Linear Transformations on Euclidean Jordan Algebras

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Abstract In this article, we study the positive principal minor (PPM) property of linear transformations on Euclidean Jordan algebras. Specifically, we give a characterization of the PPM property on the Lorentz space $L^n$ and show that the PPM property implies the $Q$ property. We also study a matrix-induced transformation on $L^n$.

Keywords Euclidean Jordan algebra · PPM property · Complementarity problems · $R_0$ property · $Q$ property · $Z$ property · GUS property · Lipschitzian GUS property

1 Introduction

A real square matrix $M$ is a P-matrix if all principal minors of $M$ are positive. Such matrices have found many applications in various fields, particularly in optimization (see [1]). There are numerous ways to describe a P-matrix, see e.g. [2, 3]. In this article, we consider the following equivalent conditions on $M \in \mathbb{R}^{n \times n}$:

(i) All principal minors of $M$ are positive.
(ii) The implication $x \in \mathbb{R}^n, x \ast Mx \leq 0 \Rightarrow x = 0$ holds, where the asterisk denotes the componentwise product.
(iii) For all $q \in \mathbb{R}^n$, the linear complementarity problem $LCP(M, q)$ has a unique solution; i.e., there exists a unique $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle x, Mx + q \rangle = 0.$$
(iv) The map \( q \mapsto \text{SOL}(M, q) \) is single valued and Lipschitzian on \( \mathbb{R}^n \), where \( \text{SOL}(M, q) \) denotes the solution set of LCP \( (M, q) \).

As can be seen, the above conditions deal with the cone \( \mathbb{R}_+^n \) (of nonnegative vectors in \( \mathbb{R}^n \)), the componentwise product \( x \ast y \), and the (usual) inner product in \( \mathbb{R}^n \). Now consider the space \( S^n \) of all \( n \times n \) real symmetric matrices with the inner product \( \langle X, Y \rangle := \text{trace}(XY) \) and Jordan product \( X \circ Y := \frac{1}{2}(XY + YX) \). In this space, we consider the cone \( S^n_+ \) of all positive semidefinite symmetric matrices. In this setting, property (ii) was extended, see [4], to a linear transformation \( L \) on \( S^n \) by means of the condition

\[
X \in S^n, \quad XL(X) = L(X)X \preceq 0 \Rightarrow X = 0
\]

where \( Z \preceq 0 \) means that \( Z \) is symmetric and negative semidefinite. It was shown in [4] that the analog of (iii) \( \Rightarrow \) (ii) holds in this setting. The above property in \( S^n \) and its nonsymmetric version were studied extensively in [5–9].

The positive principal minor property of a matrix can be extended to general linear transformations with respect to faces of a cone in the following way. For a closed convex cone \( K \) in a finite dimensional real Hilbert space \( V \), let \( \Pi_K(x) \) denote the (orthogonal) projection of \( x \in V \) onto \( K \). A closed convex cone \( F \) contained in \( K \) is said to be a face of \( K \) if the following is satisfied:

\[
x \in K, y \in K, \quad \lambda x + (1 - \lambda)y \in F, \quad \text{for some } \lambda \in (0, 1) \Rightarrow x \in F \text{ and } y \in F.
\]

Given a linear transformation \( L : V \to V \) and a face \( F \) of \( K \), the transformation \( L_{FF} : \text{Span}(F) \to \text{Span}(F) \) defined by

\[
L_{FF}(x) = \Pi_{\text{Span}(F)}(L(x)) \quad (x \in \text{Span}(F))
\]

is called a principal subtransformation of \( L \) (induced by \( F \)). We say that \( L \) has the positive principal minor property with respect to \( K \) if, for every face \( F \) of \( K \), the determinant of \( L_{FF} \) is positive. We remark that, when \( V = \mathbb{R}^n, K = \mathbb{R}^n_+, \) and \( L \) is an \( n \times n \) real matrix, the positive principal minor property coincides with the P-matrix property.

Using the above, recently, Gowda, Sznajder and Tao (see [10]) extended properties (i)–(iv) for a linear transformation defined on a Euclidean Jordan algebra and studied various interconnections between the extended concepts. It was shown in [10] that the analog of (iv) \( \Rightarrow \) (i) holds but the converse is not true in this general setting. While the above conditions are, in general, not equivalent, one may ask under which conditions they become equivalent. The main objective of this paper is to study analogs of the above properties for a linear transformation defined on a Euclidean Jordan algebra which is a finite dimensional inner product space equipped with a Jordan product, and the corresponding symmetric cone (of squares).

Outline of the paper. In Sect. 2, we cover the basic material dealing with the complementarity properties and Euclidean Jordan algebras. In Sect. 3, we give some general results on Euclidean Jordan algebras. In Sect. 4, we give the PPM property on the Lorentz cone. In Sect. 5, we study a matrix-induced transformation on \( L^n \).