TWO QUESTIONS ON THE EXTERIOR GEOMETRY OF THE PLÜCKER EMBEDDINGS OF THE GRASSMANN MANIFOLDS

A. N. Glushakov

The Plücker model of the Grassmann manifold $G^+_{p,p+q}$ is considered. The structure of intersections of $G^+_{p,p+q}$ with tangent spaces of $G^+_{p,p+q}$ regarded as subspaces of the ambient exterior algebra is described. An explicit formula for the second fundamental form of $G^+_{2,4}$ as of a hypersurface in the five-dimensional sphere is given. The level sets of the normal curvature functions for this hypersurface are studied. Bibliography: 3 titles.

§1.

The Plücker model of the Grassmann manifold $G^+_{p,p+q}$ allows us to consider $G^+_{p,p+q}$ as a submanifold of the sphere $S^{N-1}$ lying in the $N$-dimensional Euclidean space $A_p(V^{p+q})$ of $p$-vectors over the $(p+q)$-dimensional Euclidean space $V^{p+q}$, where $N = \binom{p+q}{p}$ (see [1], Sec. 6.2).

In this section, we restrict ourselves to the case $p = 2$, $q = 2$. Then $N = 6$ and the manifold $G^+_{2,4}$ is a hypersurface of the unit sphere $S^5$. We will describe the structure of level sets of the normal curvature function of $G^+_{2,4}$ in $S^5$.

The Hodge operator $* : \Lambda_2(V^4) \to \Lambda_2(V^4)$, which is orthogonal and self-adjoint, maps the space $\Lambda$ onto itself. Its eigenvalues are equal to +1 and −1, and the corresponding eigenspaces $\Lambda^+$ and $\Lambda^-$ are orthogonal, both of dimension three (see [1], Sec. 8.1). Thus, we have

$$\Lambda_2(V^4) = \Lambda^+ \oplus \Lambda^-, \quad \dim \Lambda^+ = \dim \Lambda^- = 3,$$

(1.1)

Let $e := \{e_i\}_{i=1}^4$ be an orthonormal and positively oriented basis of $V^4$. Then the bivectors $e_{ij} := e_i \wedge e_j$, $(ij) \in \Lambda(4,2)$, form a basis of $\Lambda_2(V^4)$, and we have

$${\ast} e_{12} = e_{34}, \quad {\ast} e_{13} = -e_{24}, \quad {\ast} e_{14} = e_{23},$$

$${\ast} e_{23} = e_{14}, \quad {\ast} e_{24} = -e_{13}, \quad {\ast} e_{34} = e_{12}.\tag{1.2}$$

It follows from (1.2) that the systems

$$\xi^+_1 := \frac{e_{12} + e_{34}}{\sqrt{2}}, \quad \xi^+_2 := \frac{e_{13} - e_{24}}{\sqrt{2}}, \quad \xi^+_3 := \frac{e_{14} + e_{23}}{\sqrt{2}},$$

(1.3+)

$$\xi^-_1 := \frac{e_{12} - e_{34}}{\sqrt{2}}, \quad \xi^-_2 := \frac{e_{13} + e_{24}}{\sqrt{2}}, \quad \xi^-_3 := \frac{e_{14} - e_{23}}{\sqrt{2}}.\tag{1.3-}$$

are bases of $\Lambda^+$ and $\Lambda^-$, respectively. The Grassmann manifold $G^+_{2,4}$ can be represented as the Minkowski sum of the two-spheres of radius $1/\sqrt{2}$ lying in the subspaces $\Lambda^+$ and $\Lambda^-$ (see [2]):

$$G^+_{2,4} = S^2_+(1/\sqrt{2}) + S^2_-(1/\sqrt{2}).\tag{1.4}$$

1.1. Theorem. Let $\omega \in G_{2,4}^+$. Then the intersection of the manifold $G_{2,4}^+$ with its tangent space $T_\omega G_{2,4}^+$ regarded as a linear subspace of $\Lambda_2(V^4)$ is the Clifford torus $T^2$.

Proof. Let $e$ be an orthonormal basis such that $\omega = e_{12}$. Then (see [1], Sec. 6.2) the vectors

$$e_{13}, e_{14}, e_{23}, e_{24}$$

form a basis of $T_\omega G_{2,4}^+$. Now it follows from (1.3) and (1.5) that

$$\sigma^+ := \Lambda^+ \cap T_\omega G_{2,4}^+ = \text{Lin}(\xi^+_2, \xi^+_3), \quad \sigma^- := \Lambda^- \cap T_\omega G_{2,4}^+ = \text{Lin}(\xi^-_2, \xi^-_3),$$

where $\Lambda^+$ and $\Lambda^-$ are the positive and negative parts of the exterior algebra, respectively. Then

$$T_\omega G_{2,4}^+ = \sigma^+ \oplus \sigma^-,$$

and

$$T_\omega G_{2,4}^+ \perp \omega^+, *\omega.$$  

Consider the circles

$$S^1_+(1/\sqrt{2}) := \sigma^+ \cap S^2_+(1/\sqrt{2}), \quad S^1_-(1/\sqrt{2}) := \sigma^- \cap S^2_-(1/\sqrt{2}).$$

It immediately follows from (1.4) and (1.7) that

$$T^2 = G_{2,4}^+ \cap T_\omega G_{2,4}^+ = S^1_+(1/\sqrt{2}) + S^1_-(1/\sqrt{2}).$$

Thus, the intersection considered is a flat torus canonically embedded in the Euclidean four-space. \hfill \Box

It is easy to see that torus $T^2$ lies in the three-sphere

$$S^3 = S^5 \cap T_\omega G_{2,4}^+$$

as a minimal surface.

1.2. Theorem. The flat torus $T^2$ is a totally geodesic surface in $G_{2,4}^+$.

Proof. Let $\tau \in T^2$. It follows from (1.8) that the tangent space $T_\tau T^2$ is the two-plane orthogonal to the vectors $\tau, *\tau, \omega,$ and $*\omega$ (these vectors are orthonormal). The tangent space $T_\tau G_{2,4}^+$ is the four-plane orthogonal to $\tau$ and $*\tau$. It follows that the orthogonal complement of $T_\tau T^2$ in $T_\tau G_{2,4}^+$ is the two-plane spanned by $\omega$ and $*\omega$, i.e., it is the orthogonal complement of $T_\omega G_{2,4}^+$ in $G_{2,4}^+$. Hence, for any $u, v \in T_\tau T^2$ we have $B_\tau(u, v) \in (T_\omega G_{2,4}^+)^\perp$. On the other hand, $T^2 \subset T_\omega G_{2,4}^+$, and hence $B_\tau(u, v) \in T_\omega G_{2,4}^+$. It follows that $B_\tau(u, v) = 0$ for any $u, v \in T_\tau T^2$. \hfill \Box

1.3. Let $x, y \in T_\omega G_{2,4}^+$. It is obvious that the orthogonal complement of $T_\omega G_{2,4}^+$ in $T_\omega S^5$ coincides with the one-dimensional subspace generated by the bivector $*\omega$. Therefore, $B_\omega(x, y) = \alpha * \omega$, where $\alpha \in \mathbb{R}$ should be found.

Let $\Omega : (-\varepsilon; \varepsilon) \rightarrow G_{2,4}^+$ be a smooth curve passing through $\omega$ in the direction of a vector $\tau$,

$$\Omega(0) = \omega, \quad \Omega'(0) = \tau;$$

and let $Y$ be an extension of the vector $y$ to a smooth vector field along $\Omega$ tangent to $G_{2,4}^+$:

$$Y(t) \in T_{\Omega(t)}G_{2,4}^+, \quad Y(0) = y.$$  

For each $t \in (-\varepsilon; \varepsilon)$ we have

$$\langle Y(t), *\Omega(t) \rangle = 0.$$

Differentiating this relation and taking the value of the derivative at $t = 0$, we obtain

$$\langle Y'(0), *\omega \rangle + \langle y, *\tau \rangle = 0, \quad \langle Y'(0), *\omega \rangle = -\langle y, *\tau \rangle.$$

Note that the left-hand side of the second relation is equal to $\alpha$, and hence

$$B_\omega(x, y) = -\langle *\omega, y \rangle * \omega. \quad (1.12)$$

1.4. Let us orient the hypersurface $G_{2,4}^+ \subset S^5$ by taking the bivector $*\omega$ as the normal at the point $\omega \in G_{2,4}^+$. The normal curvature of the surface $G_{2,4}^+$ in the direction $x \in S^3$ (see (1.11) and (1.12)) is

$$k_n(x) = \langle B_\omega(x, x), *\omega \rangle = -\langle *\omega, x \rangle.$$  

Consider a level set of the function $k_n(x)$,

$$K_c := \{x \in S^3 | k_n(x) = c\}, \quad c \in \mathbb{R}. \quad (1.14)$$