TOPOLOGY AND A LORENTZ-ININVARIANT PSEUDO-RIEMANNIAN METRIC OF THE MANIFOLD OF DIRECTIONS IN THE PHYSICAL SPACE

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In the mathematical model of the special relativity theory, a two-dimensional Minkowski subspace is treated as a one-dimensional direction in the physical space. The manifold of such planes is naturally endowed with the structure of a pseudo-Riemannian manifold on which the group of isochronous Lorentz transformations acts transitively by isometries. In this paper, the topology and the metric geometry of this manifold are studied.

Bibliography: 4 titles.

§1. INTRODUCTION

1.1. In the four-dimensional Minkowski point space $M$, every time-like curve

$$x = x(t), \quad (x'(t))^2 > 0,$$

is treated as a particle with real and positive mass (see [1], p. 12). With an inertial particle $x$ ($x'(t) = a = \text{const}, \ a^2 > 0$), we can associate its physical space $M(x) := M/\text{Lin}(x')$. Every two-dimensional plane $V$ containing an inertial particle $x$ determines a line $\ell = \text{pr}_x(V)$ in the physical space of this particle. Here the mapping

$$\text{pr}_x : M \rightarrow M(x)$$

is the projection of the space $M$ to its factor space. Therefore, we say (see [1], p. 13) that a two-dimensional plane containing a line of real length determines a one-dimensional direction in the physical space.

1.2. Any inertial particle $\bar{x}$ belonging to the plane $V$ and different from $x$ has a uniquely determined projection onto the line $\ell$. If we fix a time-like orientation in the Minkowski vector space $M_0$ (see [2], §8), then the inertial particle $\bar{x}$ obtains a natural time-like orientation. This orientation is transferred to the line $\ell$ by the projection mapping (1.2). We see that fixing an orientation on the plane $U$ is equivalent to the choice of direction of the line $\ell$ (under a fixed time-like orientation of the space $M_0$). It is natural to represent the class of planes parallel to $V$ and oriented similarly to $V$ by a two-dimensional subspace $V_0$ of the space $M_0$ (with the corresponding orientation). This plane $V_0$ contains a time-like direction, hence it is a two-dimensional Minkowski space. We denote by $G^1$ the set of these planes. Let $G^2$ denote the set of space-like planes, and let $G^0$ denote the set of isometric planes. In this case, the Grassmann manifold $G^+_2(M_0)$ of oriented planes can be represented as the disjoint union

$$G^+_2(M_0) = G^0 \cup G^1 \cup G^2.$$

Observe that the subsets $G^1$ and $G^2$ are open in the manifold $G^+_2(M_0)$. We show in Sec. 2.4 that $G^0$ is a closed three-dimensional submanifold of $G^+_2(M_0)$.

Thus, we can treat the manifold $G^1$ as the manifold of one-dimensional directions in the physical space.

In the present paper, we investigate the topology and the metric geometry of the manifold $G^1$.

§2. TOPOLOGY OF THE MANIFOLDS $G^0$, $G^1$, AND $G^2$

2.1. The group $L^+_+$ of bounded Lorentz transformations acts transitively on the manifolds $G^i$. Recall that the family of bounded Lorentz transformations is the component of unity in the Lorentz group $L$. Consider an oriented plane $\alpha \in G^i$. Denote by $H_\alpha$ its stationary subgroup. Then we have a diffeomorphism

$$G^i \cong L^+_+ / H_\alpha$$

(see [4], §30).

2.2. The manifolds \( G^1 \) and \( G^2 \) are diffeomorphic. This diffeomorphism is realized by the passage to the orthogonal complement:

\[
\perp: G^1 \to G^2, \quad \alpha \mapsto \alpha^\perp.
\]  

(2.2)

It follows from this definition that the orientation of the plane \( \alpha^\perp \), together with the orientation of \( \alpha \), determines the positive orientation of the Minkowski space \( M_0 \). The stationary subgroup \( H_\alpha \) is isometric to the product of the group \( L^+_\perp(\alpha) \) of hyperbolic rotations of the plane \( \alpha \) and the group \( SO(\alpha^\perp) \) of orthogonal rotations of the plane \( \alpha^\perp \) ([2], §14). Hence, we see from (2.1) that

\[
G^1 \simeq G^2 \simeq L^+\perp_\perp/(L^+\perp_\perp(\alpha) \times SO(\alpha^\perp)) \simeq SO(1,4)/(R^1 \times S^1).
\]  

(2.3)

2.3. Every coset \([xH], x \in \Lambda_\perp \), on the right in (2.1) corresponds in a one-to-one manner to a stationary subgroup \( H_\tilde{\alpha} = x^{-1}H_\alpha x \) of the oriented plane \( \tilde{\alpha} = x^{-1}(\alpha) \). In the Cayley–Klein model of the three-dimensional Lobachevsky space (see [2]), every two-dimensional oriented Minkowski plane in the space \( M_0 \) corresponds in a one-to-one manner to a directed chord of a unit Euclidean disk \( D \). The set of such chords is represented by an ordered pair of points on the sphere \( S^2 = \partial D \). Hence

\[
G^1 \simeq G^2 \simeq (S^2 \times S^2) \setminus \text{diag}(S^2 \times S^2).
\]  

(2.4)

It follows from (2.4) that the manifold \( G^1 \) is connected.

2.4. Consider an oriented plane \( \alpha \in G^0 \). This plane corresponds in a one-to-one manner to a pair \( x, \ell \), where \( x \in S^2 \) (the sphere \( S^2 \) is introduced in Sec. 2.3) and \( \ell \) is a directed line in the tangent plane \( T_xS^2 \). Therefore,

\[
G^0 \simeq S^1(S^2) \subset TS^2,
\]  

(2.5)

where \( S^1(S^2) \) denotes the spherical bundle of circles over the two-dimensional sphere.

§3. Lorentz-invariant scalar product on the exterior algebra \( \Lambda(M_0^{n+1}) \)

3.1. One can canonically extend the scalar product of the Minkowski space \( M_0^{n+1} \) to the whole of the exterior algebra \( \Lambda(M_0^{n+1}) \) (see [3], Sec. 1.5). In this case, the Lorentz group \( L \) acts on this algebra by isometries (see (2.3)). Any Minkowski tetrad \( e \),

\[
e = \{e_0, e_i\}_{i=1}^n, \quad e_0^2 = -e_i^2 = 1, \quad \langle e_0, e_i \rangle = \langle e_i, e_j \rangle = 0, \quad i \neq j.
\]  

(3.1)

corresponds to a basis \( \{e_\lambda\} \),

\[
e_\lambda = e_{i_1} \wedge \ldots \wedge e_{i_p},
\]  

(3.2)

where \( \lambda = (i_1, \ldots, i_p) \in \Lambda(n+1, p) = \{(i_1, \ldots, i_p) \mid 0 \leq i_1 < \ldots < i_p \leq n\} \),

of the space \( \Lambda_p(M_0) \) of \( p \)-vectors. Since for any simple polyvector \( \omega = f_1 \wedge \ldots \wedge f_p \) we have

\[
\omega^2 = \langle \omega, \omega \rangle = \det(\langle f_i, f_j \rangle),
\]  

(3.3)

it follows that the basis \( \{e_\lambda\} \) above is orthonormal. For \( n = 3 \) and \( p = 2 \), we have

\[
e_{01}^2 = e_{02}^2 = e_{03}^2 = -e_{12}^2 = -e_{13}^2 = -e_{23}^2 = 1, \quad \langle e_\lambda, e_\mu \rangle = 0, \quad \lambda \neq \mu.
\]  

(3.4)

Thus, the space \( \Lambda_2(M_0^4) \) of bivectors is a pseudo-Euclidean space with signature \((+++--)\).