Some approximation problems for functions of class $L_p$ are considered. Bibliography: 4 titles.

1. Introduction

1. Introduce the notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers. Integrals written as

$$\int_{-\infty}^{+\infty} f$$

are understood as improper integrals, i.e.,

$$\int_{-\infty}^{+\infty} f = \lim_{b \to +\infty} \lim_{a \to -\infty} \int_{a}^{b} f,$$

where $f$ is summable on any finite interval. Otherwise, the integral should be regarded as the Lebesgue integral. By definition,

$$\sum_{k=-\infty}^{+\infty} = \sum_{k \in \mathbb{Z}} = \lim_{n \to +\infty} \sum_{k=m}^{n}$$

We denote by $C$ the space of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the norm

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$$

and by $V$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ with finite variation.
\[ \begin{align*} 
\sup_{-\infty}^{+\infty} V(f) &< +\infty, \\
CV = C \cap V, & f(\pm \infty) = \lim_{x \to \pm \infty} f(x). 
\end{align*} \]

The modulus of continuity \( \omega(f, h) \) of a function \( f \in C \) is defined by the formula

\[
\omega(f, h) = \sup_{|t| \leq h} \| f(\cdot + t) - f(\cdot) \|,
\]

The set \( E \) consists of functions \( f \in CV \) such that

\[
\sup_{f \in E} \| f \| < +\infty, \quad \sup_{f \in E} V f < +\infty, \quad \lim_{h \to 0} \sup_{f \in E} \omega(f, h) = 0,
\]

and the set \( A \) is formed by functions \( D : \mathbb{R} \to \mathbb{R} \) such that

\[
\int_{-\infty}^{+\infty} D = 1.
\]

The following assertion is proved in [1].

**Theorem A.** Let \( D \in A, \alpha, h > 0, x, u \in \mathbb{R} \),

\[
\phi(u) = \int_{u}^{+\infty} D(t) dt.
\]

Then for \( f \in V \) the series

\[
V_{\alpha, h}(f, x) = f(-\infty) + \sum_{k \in \mathbb{Z}} (f((k+1)h) - f(kh)) \int_{0}^{1} \phi \left( \frac{th + kh - x}{\alpha} \right) dt
\]

uniformly converges with respect to \( x \in \mathbb{R} \) and the following equality holds:

\[
\lim_{\alpha, h \to 0} \sup_{f \in E} \| V_{\alpha, h}(f) - f \| = 0.
\]

In this paper, we establish similar assertions in the spaces \( L_p(\mathbb{R}) \), \( p \geq 1 \).

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2. Notation and Auxiliary Assertions

We denote by \( L_p(\mathbb{R}) \), \( p \geq 1 \), the space of \( p^{th} \)-power summable functions \( f : \mathbb{R} \to \mathbb{R} \) equipped with the norm

\[
\| f \|_p = \left( \int_{\mathbb{R}} |f|^p \right)^{\frac{1}{p}}.
\]