SOME APPLICATIONS OF THE DUHAMEL PRODUCT

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The Duhamel product of functions \( f \) and \( g \) is defined by the formula

\[
(f \circ g)(x) = \frac{d}{dx} \int_0^x f(x - t)g(t)dt.
\]

In the present paper, the Duhamel product is used in the study of spectral multiplicity for direct sums of operators and in the description of cyclic vectors of the restriction of the integration operator \( f(x, y) \mapsto \int_0^x \int_0^y f(t, \tau)d\tau dt \) in two variables to its invariant subspace consisting of functions that depend only on the product \( xy \). Bibliography: 13 titles.

INTRODUCTION

1. Let \( \text{Hol}(\mathbb{D}) \) be the space of functions that are holomorphic in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \); we consider this space with the topology of compact convergence. In the space \( \text{Hol}(\mathbb{D}) \), the Duhamel product is defined as the derivative of the Mikusinski convolution:

\[
(f \circ g)(z) = \frac{d}{dz} \int_0^z f(z - t)g(t)dt = \int_0^z f'(z - t)g(t)dt + f(0)g(z),
\]

where the integrals are taken over the segment joining the points 0 and \( z \) (see [1]).

The Duhamel product is widely applied in various domains of calculus, for example, in the theory of differential equations with constant coefficients and in solution of some boundary-value problems of mathematical physics; the real-valued analog of this product plays an important role in the Mikusinski operator calculus (see [1–3]). In the work [1], Wigley applied the Duhamel product for the first time to describe closed ideals of the algebra \( \text{Hol}(\mathbb{D}) \). In the author’s works [4, 5], the product was applied in the description of cyclic vectors of the integration operator \( J, (Jf)(x) = \int_0^x f(t)dt \), for some function spaces and in the proof of the unicellularity of the integration operator in the Banach space of functions holomorphic in \( \mathbb{D} \). See [6–8] for other applications of the Duhamel product.

2. In Sec. 2 of this work, we apply the Duhamel product in the study of multiplicity of spectra for direct sums of operators. In Sec. 3, we apply an analog of the Duhamel product for functions of two variables to describe cyclic vectors of the restriction of the double integration operator,

\[
f(x, y) \mapsto \int_0^x \int_0^y f(t, \tau)d\tau dt,
\]

to an invariant subspace of a special form.

Some of the results of this work have been obtained long ago; they are contained in the author’s thesis [9]. The author decided to publish these results in a journal paper, in particular, since new applications were found.

3. We use more or less standard notation. We denote by \( L(X) \) the algebra of bounded linear operators in a Banach space \( X \), \( \text{Lat} A \) denotes the lattice of invariant subspaces of an operator \( A \), and \( \text{span} \{ x_i \} \) is the closed linear hull of vectors \( x_i \).

Recall that a subspace \( E \subset X \) is called a cyclic (generating) subspace for an operator \( A \in L(X) \) if \( \text{span} \{ A^n E : n \geq 0 \} = X. \) A vector \( x \in X \) is called cyclic if
\[
\text{span} \{ A^n x : n \geq 0 \} = X.
\]
We denote by \( \text{Cyc} (A) \) the set of cyclic vectors of an operator \( A. \) The multiplicity of the spectrum of an operator \( A \) is
\[
\mu(A) \overset{\text{def}}{=} \inf \{ \dim E : \text{span} \{ A^n E : n \geq 0 \} = X \}
\]
(or the symbol \( \infty \)). An operator \( A \in L(X) \) is called cyclic if \( \mu(A) = 1; \) an operator \( A \) is called unicellular if the lattice \( \text{Lat} A \) is linearly ordered with respect to inclusions.

§1. DUHAMEL PRODUCT AND SOME OF ITS PROPERTIES

Let \( \text{Hol}(D) \) be the space of functions that are holomorphic in the unit disk \( D; \) we consider this space with the topology of uniform convergence on compact subsets of \( D. \) For two functions \( f(z) = \sum_{n \geq 0} \hat{f}(n)z^n \) and \( g(z) = \sum_{n \geq 0} \hat{g}(n)z^n \in \text{Hol}(D) \) (where \( \hat{f}(n) = \frac{f^{(n)}(0)}{n!} \) is the \( n \)th Taylor coefficient of a function \( f \)), the Duhamel product is defined as follows (see [1]):
\[
(f \circ g)(z) \overset{\text{def}}{=} \frac{d}{dz} \int_0^z (f(z - t)g(t)dt,
\]
where the integral is taken over the segment joining the points 0 and \( z. \) It is easy to see that the Duhamel product satisfies all the axioms of multiplication, \( \text{Hol}(D) \) is an algebra with respect to \( \circ \) as well, and the function \( f(z) \equiv 1 \) is the unit element of the algebra \( \text{Hol}(D), \circ). \)

Let \( B \) be the Borel transformation acting from \( \text{Hol}(D) \) into the space \( C[[Z]] \) of formal power series over the field \( C \) of complex numbers; this transformation is defined by the following formula:
\[
B\left( \sum_{n \geq 0} \hat{f}(n)z^n \right) \overset{\text{def}}{=} \sum_{n \geq 0} n! \hat{f}(n)Z^n.
\]
The inverse Borel transformation \( B^{-1} \) acts by the following formula:
\[
B^{-1}\left( \sum_{n \geq 0} a_n Z^n \right) \overset{\text{def}}{=} \sum_{n \geq 0} \frac{a_n}{n!} z^n.
\]
The following simple but useful statement holds.

**Lemma 1.** Let \( f, g \in \text{Hol}(D). \) The following equalities are valid for the product \( \circ \) in \( \text{Hol}(D): \)

(a) \( f \circ g = \sum_{n \geq 0} \sum_{k=0}^n \hat{f}(k)\hat{g}(n-k)k! (n-k)! \frac{1}{n!} z^n; \)

(b) \( f \circ g = (Bf)(J)g = (Bg)(J)f, \) where \( J \) is the integration operator in \( \text{Hol}(D) \) and \( (Bf)(J)g \overset{\text{def}}{=} \sum_{n \geq 0} n! \hat{f}(n)(J^n g)(z); \)

(c) \( f \circ g = B^{-1}(Bf \cdot Bg). \)

**Proof.** Proofs of statements (a) and (c) can be found in [1]; let us prove item (b). First we show that the series \( \sum_{n \geq 0} n! \hat{f}(n)(J^n g)(z) \) defines a function in \( \text{Hol}(D). \) For this purpose, we show that the series above converges uniformly on any compact subset of the disk \( D. \)

Fix a number \( r, 0 < r < 1 \). It is enough to show that the series converges absolutely and uniformly on the disk \( |z| < r < 1. \)