DUALITIES BETWEEN ALMOST COMPLETELY DECOMPOSABLE GROUPS AND THEIR ENDOMORPHISM RINGS

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Abstract. We prove that endomorphism rings of nearly isomorphic, almost completely decomposable groups of ring type are also nearly isomorphic as additive structures. On this basis, acd groups can be considered in a dual connection with their endomorphism rings.

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1. Introduction

The class of almost completely decomposable groups admits a wide variety of nonisomorphic direct decompositions (see [2–4, 7–10, 14, 15, 17]). Traditionally, Butler groups (in particular, almost completely decomposable groups) and their direct decompositions are classified up to near isomorphisms, i.e., equivalences, which are weaker than isomorphisms but preserve decomposability properties sufficiently precisely (see [1, Theorem 7.16]). Direct decomposition properties of acd groups are reflected in (or determined by) their endomorphism rings, which were studied by Mader and Schultz [16]. A tight connection between acd groups and their endomorphism rings was established in the case of block-rigid crq groups [6]. For a wider class of acd groups, we prove the following fact: if two acd groups of ring type are nearly isomorphic, then their endomorphism rings are also nearly isomorphic as Abelian groups. This implies that direct decompositions of endomorphism rings of nearly isomorphic groups into one-sided ideals are closely connected with each other and also with decompositions of the groups [12, Chap. XV, 106].

The monograph of Mader [15] together with the classical work of Fuchs [12] serve as the main references.

For a group generated by a system of elements, we use the symbol \( \langle \ldots \rangle \); the rank of a group \( X \) is denoted by \( \text{rk} \, X \). As usual, \( V \subset X \) means that \( V \) is a subgroup of \( X \) and

\[
V_* = \{ g \in X : \exists n \in \mathbb{N} \text{ such that } ng \in V \}
\]

denotes the purification of \( V \) in \( X \). The type \( \text{tp}_X \, g \) of an element \( g \in X \) (\( g \neq 0 \)) can be defined as the isomorphism class of the rational group \( \tau \), which is isomorphic to \( \langle g \rangle_* \) in \( X \) and contains \( \mathbb{Z} \). Then we can say that the element \( g \) is of type \( \tau, \mathbb{Z} \subset \tau \subset \mathbb{Q} \). Furthermore, \( \text{tp}_X \, g \) coincides with the group type, \( \text{tp} \, X \), if \( X \) is a homogeneous group. We write \( \tau(p) = \infty \) or \( p\tau = \tau \) if \( 1/p^n \) belongs to \( \tau \) for any natural number \( n \) (\( p

is a prime). Following standard definitions, also \( X^*(\tau) = \sum_{\sigma \succ \tau} X(\sigma) \) and \( X^\sharp(\tau) \) is the purification of \( X^*(\tau) \) in \( X \). A type \( \tau \) is critical, or a member of \( T_{CR}(X) \), if \( X(\tau)/X^\sharp(\tau) \neq 0 \) (see [15, p. 37, Definition 2.4.6]).

Now we discuss the objects under investigation. An acd group \( X \) (almost completely decomposable group) is a torsion-free Abelian group of finite rank that contains a completely decomposable group \( A \) for which \( X/A \) is a finite group. If, in addition, \( X/A \) is a cyclic group, then \( X \) is called a crq-group (i.e., an acd group with cyclic regulator quotient). \( X \) is called a group of ring type if \( T_{CR}(X) \) consists only of idempotent types (i.e., the types which are types of idempotent rational groups or, in other words, can be represented by characteristics consisting only of 0’s and \( \infty \)’s (see [15, p. 13] and [12, Sec. 85]). The group \( A \) decomposes uniquely up to isomorphism into its \( \tau \)-homogeneous components \( A_\tau \), i.e., direct sums of rank-1 groups of type \( \tau \), \( \tau \in T_{CR}(X) \) \( (A_\tau = 0 \text{ if } \tau \notin T_{CR}(X)) \). We say that \( X \) is a block-rigid group if \( T_{CR}(X) \) is an antichain. If, in addition, \( \text{rk} A_\tau = 1 \) for all \( \tau \in T_{CR}(X) \), then \( A \) and \( X \) are called rigid groups.

The Baer–Kaplansky theorem was proved in [6] for block-rigid crq-groups up to near-isomorphism. We showed that two such groups \( X \) and \( Y \) of ring type are nearly isomorphic if and only if \( \text{End} X \cong \text{End} Y \). In particular, it was proved that if \( X \) is a rigid group from this class, then \( X \) is nearly isomorphic to \( \text{End} X^+ \) with respect to the sum operation. Now we study relationships between acd groups and their endomorphism rings in a more general situation.

Any acd group \( X \) has a distinguished, completely decomposable subgroup \( R(X) \) isomorphic to \( A \). This group \( R(X) \) is the regulator of \( X \) and \( [X : R(X)] \) is the regulator index of \( X \). Its regulator exponent is the exponent \( e =: \exp X/R(X) \) of the regulator quotient \( X/R(X) \). The regulator \( R(X) \) is a fully invariant subgroup of \( X \).

Our notation is standard and can be found in [6, 7, 12, 15]. If a group \( X \) is isomorphic to \( Y \), we write \( X \cong Y \); a near-isomorphism of these groups is denoted by \( X \cong_{nr} Y \). As usual, \( \mathbb{Z} \) is the group of all integers, \( \mathbb{N} \) is the set of all natural numbers, and \( \mathbb{Q} \) is the additive group of rational numbers.

If an integer \( q \) is divisible by an integer \( p \), we write \( p \mid q \). The symbol \( |c| \) denotes the order of a group element \( c \in X \), and \( |C| \) is the cardinality of a group \( C \) (it is used only for finite groups and sets).

To use linear-algebraic tools, we introduce the divisible hull \( D_H \) of a finite-rank, torsion-free, Abelian group \( H \). The notation \( E^+ \) is used for the additive group of a ring \( E \). We write \( f \in \text{Mon}(G, F) \) if \( f : G \to F \) is an injective homomorphism.

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2. Preliminaries

We consider a class \( \mathcal{A} \) of almost completely decomposable groups \( X \) of ring type having the regulator

\[
A = \bigoplus_{\tau \in T} A_\tau \cong \bigoplus_{\tau \in T} n_{\tau} \tau \quad \text{with} \quad T = T_{CR}(A),
\]

(2.1)