INEQUALITIES FOR MAJORIZING ANALYTIC FUNCTIONS

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For analytic functions satisfying Meyman’s majorization conditions, with the use of classical properties of conformal mappings exact inequalities complementing and strengthening the results of Akhiezer and Meyman are derived. As a corollary, for the modulus of the derivative of a rationally-trigonometric function a Bernstein type bound, which implies the result by Borwein, Erdélyi, and Zhang, is obtained. Bibliography: 10 titles.

1. Introduction

By admissible functions are meant analytic functions univalent on the entire finite plane with isolated singular points. By $E$ is denoted the set of all admissible functions $\omega(z)$ satisfying the condition

$$\left|\frac{\omega(\overline{z})}{\omega(z)}\right| < 1, \quad \text{Im } z > 0.$$ 

A function with real values at the real points of its domain of definition will be referred to as a real one. An admissible real function $f(z)$ will be said to be subordinate to a function $\omega(z) \in E$ if

$$\left|\frac{f(z)}{\omega(z)}\right| < 1, \quad \text{Im } z > 0.$$ 

In [1], Meyman proved that these conditions imply the validity of the inequality $|f'(x)| \leq |\omega'(x)|$ on the real axis. In the present paper, first we prove exact inequalities of the following nature. For every $z, \text{Im } z > 0$, a domain in the unit disk is indicated that contains the value $f(z)/\omega(z)$ and is bounded by ellipses with focuses at the points $\pm \sqrt{\omega(z)}/\omega(z)$, where $\overline{\omega(z)} = \omega(\overline{z})$ (Theorem 1). Theorem 1 strengthens the Akhiezer inequality [2, p. 343]

$$2|f(z)| \leq |\omega(z)| + |\overline{\omega(z)}|, \quad z \in \mathbb{C},$$

established under much less general assumptions. At the zeros of $\overline{\omega(z)}/\omega(z)$, Theorem 1 yields

$$\left|\frac{f(z)}{\omega(z)}\right| \leq \frac{1}{2}.$$ 

A stronger result for such points is provided by Theorem 3. Theorem 2 yields a bound for the derivative $f'(x)$ at points of the real axis. As implications, inequalities strengthening similar results by Meyman are presented. The application of the main results of the paper to specially deviced analytic functions leads to a series of inequalities for polynomials and rational functions. Corollary 5 improves the result by Dubinin [3, p. 72], extends and strengthens the result by Borwein, Erdélyi, and Zhang [4] for rationally-trigonometric functions, which, in its turn, implies the classical Bernstein–Szegő inequality. The paper also improves the known result by Stechkin [5] for trigonometric polynomials.

Section 2 introduces a number of auxiliary functions, including a conformal and univalent mapping $w = g(\zeta)$ constructed from given functions $\omega(z)$ and $f(z)$, which is of key importance for the paper. Section 3 presents the main results, which are obtained by applying results of the geometric theory of functions of a complex variable to the above conformal mapping. This approach to deriving inequalities for polynomials was suggested by Dubinin in [3, 6, 7].

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2. Construction of a conformal mapping

Let a function \( f(z) \) be subordinate to a function \( \omega(z) = s(z) + it(z) \in E \), where \( s(z) \) and \( t(z) \) are the admissible real function defined via the equalities
\[
s(z) = \frac{\omega(z) + \overline{\omega}(z)}{2}, \quad t(z) = -i \frac{\omega(z) - \overline{\omega}(z)}{2};
\]
\( \gamma \) is the set of zeros of the fraction \( \overline{\omega}(z)/\omega(z) \); \( \gamma \subset \gamma \) are the zeros of multiplicity one. Further, let
\[
H(z) = \overline{\omega}(z)\omega(z) = s^2(z) + t^2(z), \quad \rho(z) = f(z)/\sqrt{H(z)}, \quad \delta(z) = s(z)/\sqrt{H(z)}.
\]

By \( w = \Phi(z) \) denote the function that maps, conformally and univalently, the exterior of the interval \([-1, 1]\) onto the domain \(|w| > 1\) in such a way that \( \Phi(\infty) = \infty \) and \( \Phi(1) = 1 \). Then \( w = \Phi(z) \) is the function inverse to the Zhukovsky function
\[
z = \frac{1}{2} \left( w + \frac{1}{w} \right), \quad |w| > 1.
\]
The representation \( \Phi(z) = z + \sqrt{z^2 - 1} \) implies that
\[
\Phi'(z)/\Phi(z) = 1/\sqrt{z^2 - 1}, \quad z \neq \gamma.
\]
Everywhere in the sequel, we assume that \( \gamma \neq \emptyset \).
Assume that the condition
\[
\lambda(a) := 2 \lim_{z \to a} \frac{f(z)}{\omega(z)} = \lim_{z \to a} \frac{f(z)}{s(z)} \neq 0
\]
is fulfilled for a certain \( a \in \gamma \). Let
\[
z(w) = \frac{\overline{\omega}w + a}{w + 1}
\]
be the conformal and univalent mapping of the disk \(|w| < 1\) onto the upper half-plane such that \( z(0) = a \), and let \( w = w(z) \) be the function inverse to \( z(w) \).
Assuming that in the definitions of the two-valued functions \( \rho(z) \) and \( \delta(z) \) the values of \( \sqrt{H(z)} \) coincide, on the set
\[
G_z := \{ z : \text{Im} z > 0, \rho(z) \notin [-1, 1] \}
\]
define the function
\[
L(z) := \frac{\Phi[\rho(z)]}{\Phi[\delta(z)]} = \frac{f(z) + \sqrt{f^2(z) - H(z)}}{\omega(z)},
\]
which, in view of the oddness of \( \Phi(z) \), is univalent and analytic. By the generalized maximum modulus principle [8, p. 250],
\[
|L(z)| \leq 1, \quad z \in G_z.
\]
(3)
Introduce into consideration the function
\[
\zeta = F(w) = wL^{-k(a)[z(w)]}, \quad w \in G_w = w(G_z),
\]
where
\[
k(a) = \begin{cases} 1 & \text{if } a \in \gamma \setminus \gamma_1, \\ 2 & \text{if } a \in \gamma_1. \end{cases}
\]
Let \( \mathcal{D} \) be the collection of the domains constituting the set
\[
G_w \setminus \{ w : |F(w)| = 1 \}.
\]
As the point \( w \) approaches the boundary of each of the domains in \( \mathcal{D} \), all the limit values of \( |F(w)| \) are no less than 1. This follows from inequality (3), the inequality
\[
|w\Phi^{k(a)[\delta(z)]}|^{-1} = \frac{z - \overline{\omega}}{z - a} \left| \frac{\overline{\omega}(z)}{\omega(z)} \right|^{k(a)/2} \leq 1, \quad z = z(w), \quad |w| < 1,
\]
(4)