LOWER SEMICONTINUITY OF SOME FUNCTIONALS UNDER PDE CONSTRAINTS: AN $A$-QUASICONVEX PAIR

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The problem of establishing necessary and sufficient conditions for l.s.c. under PDE constraints is studied for a special class of functionals:

$$(u, v, \chi) \mapsto \int_{\Omega} \left\{ \chi(x) \cdot F^+(x, u(x), v(x)) + (1 - \chi(x)) \cdot F^-(x, u(x), v(x)) \right\} dx,$$

with respect to the convergence $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^p(\Omega; \mathbb{R}^d)$, $A v_n \to 0$ in $W^{-1,p}(\Omega)$, and $\chi_n \rightharpoonup \chi$ in $L^p(\Omega)$, where $\chi_n \in Z := \{ \chi \in L^\infty(\Omega) : 0 \leq \chi(x) \leq 1$ for a.e. $x \}$.

$A v = \sum_{i=1}^{N} A(i) \frac{\partial v}{\partial x^i}$ is a constant-rank partial differential operator. The main result is that if the characteristic cone of $A$ has the full dimension, then the l.s.c. is equivalent to the fact that the $F^\pm$ are both $A$-quasiconvex and $F^+(x, u, \cdot) - F^-(x, u, \cdot) \equiv C(x, u)$ for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^d$.

As a corollary, we obtain several results for the functional

$$(u, v, \chi) \mapsto \int_{\Omega} \chi(x) \cdot f(x, u(x), v(x)) dx$$

with respect to the same convergence. We show that this functional is l.s.c. iff $f(x, u, v) \equiv g(x, u)$.

Bibliography: 14 titles.

1. Introduction and statement of the problem

The property of sequential weak lower semicontinuity of a functional $I[u]$ defined on a Banach space $X$:

$$I[u] \leq \liminf_{n \to \infty} I[u_n] \quad \text{for every sequence} \quad u_n \rightharpoonup u,$$

plays a crucial role for applying Direct Methods of the Calculus of Variations. The existence of minimizers of the functional $I[u]$ can be proved by exploiting this property; this property turns out to be useful for studying some properties of minimizers. There has been an extensive research on this topic; we refer, for example, to the works [1, 3, 4, 8, 11, 12].

We treat the following problem: establish a necessary and sufficient condition for the lower semicontinuity of the functional

$$I[u, v, \chi] = \int_{\Omega} g(x, u(x), v(x), \chi(x)) dx,$$

where $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ has the following special form:

$$g(x, u, v, \chi) = \chi \cdot F^+(x, u, v) + (1 - \chi) \cdot F^-(x, u, v);$$

thus,

$$I[u, v, \chi] = \int_{\Omega} \left\{ \chi(x) \cdot F^+(x, u(x), v(x)) + (1 - \chi(x)) \cdot F^-(x, u(x), v(x)) \right\} dx,$$


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where $F^{\pm} : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ are Carathéodory functions and $\Omega \subset \mathbb{R}^N$ is a bounded domain, with respect to the following convergence:

\[
    u_n \to u \text{ in measure}, \quad Z \ni \chi_n \to \chi \text{ in } L_p(\Omega), \quad v_n \to v \text{ in } L_p(\Omega; \mathbb{R}^d), \quad \text{and } Av_n \to 0 \text{ in } W^{-1,p}(\Omega),
\]

where $Z = \{ \chi \in L_\infty(\Omega) : 0 \leq \chi(x) \leq 1, \text{a.e. } x \}$. Here

\[
    Av = \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}
\]

is a constant-rank partial differential operator, where $A^{(i)} \in \text{Lin}(\mathbb{R}^d; \mathbb{R})$.

Functionals of type (1) find their application, for instance, in mechanics for describing the nonregularized energy of a two-phase medium in the case of zero surface tension (see, for example, [12, 13]) and in the optimal design of thin films (see [2, 9]).

2. Main results

Following [8], we introduce the following notation:

\[
    A(w) = \sum_{i=1}^{N} A^{(i)} w_i \in \text{Lin}(\mathbb{R}^d; \mathbb{R}), \quad w \in \mathbb{R}^N.
\]

Recall that the operator $A$ defined in (4) is called a constant-rank partial differential operator if there exists $r \in \mathbb{N}$ such that

\[
    \text{rank } A(w) = r \quad \text{for all } w \in S^{N-1}.
\]

In the sequel, we always assume that $A$ is a constant-rank partial differential operator. The characteristic cone of the operator $A$ is defined as follows (see also [14]):

\[
    \Lambda = \bigcup_{w \in S^{N-1}} \ker A(w).
\]

Recall that the work [8] contains a classical result which states a necessary and sufficient condition for the lower semicontinuity of the functional

\[
    I_0[u, v] = \int_{\Omega} f(x, u(x), v(x)) \, dx
\]

under suitable growth conditions imposed on $f$ and the constant rank assumption, with respect to the following convergence:

\[
    u_n \to u \text{ in measure}, \quad v_n \to v \text{ in } L_p(\Omega; \mathbb{R}^d), \quad \text{and } Av_n \to 0 \text{ in } W^{-1,p}(\Omega).
\]

A function $f$ satisfying this condition is called $A$–quasiconvex. Denote by $T_N$ the torus $\mathbb{R}^N/\mathbb{Z}^N$ and by $C^\infty(T_N; \mathbb{R}^d)$ the space of smooth periodic functions.

**Definition 2.1.** A function $f : \mathbb{R}^d \to \mathbb{R}$ is called $A$–quasiconvex if

\[
    f(v) \leq \int_{T_N} f(v + w(x)) \, dx
\]

for all $v \in \mathbb{R}^d$ and $w \in C^\infty(T_N; \mathbb{R}^d)$ such that $A(w) = 0$ and $\int_{T_N} w(x) \, dx = 0$.

Let us introduce the following definition.