RETRACT EXTENSIONS OF ORDERED SETS

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The concept of retract extension of an ordered set is introduced and studied. We give an example of a retract extension and an example that is not a retract extension. Bibliography: 12 titles.

1. Introduction. Prerequisites

The extension problem for groups is as follows: given two groups $H$ and $K$, construct all groups $G$ that have a normal subgroup $N$ such that $N$ is isomorphic to $H$ (in symbol, $N \cong H$) and $G/N \cong K$ (where $G/N$ is the quotient of $G$ by $N$). The group $G$ is called the Schreier extension or simply the extension of $H$ by $K$. Given a semigroup $S$ and a semigroup $T$ with zero, a semigroup $V$ is called an (ideal) extension (or simply an extension) of $P$ by $Q$ if there exists an ideal $P'$ of $V$ such that $P'$ is isomorphic to $P$ and the Rees quotient $V/P'$ is isomorphic to $Q$. The extension problem for semigroups (or ordered semigroups) is as follows. Given a semigroup $S$ and a semigroup $T$ with zero ($S$ and $T$ are disjoint), construct all the semigroups $V$ that are extensions of $S$ by $T$. For the definition of the Rees quotient for semigroups and ordered semigroups, we refer to [12] and [9], respectively. Ideal extensions of semigroups were considered in [3] with a detailed exposition of the theory presented in [4, 12]. Extensions of weakly reductive semigroups, strict and pure extensions, retract extensions, dense extensions, equivalent extensions were also considered in [12]. Ideal extensions of totally ordered semigroups are studied in [6, 7] and ideal extensions of topological semigroups are dealt with in [2, 5]. We refer to [8] for ideal extensions of lattices and to [9] for ideal extensions of ordered semigroups. Inspired by semigroups, ideal extensions of partially ordered sets were studied in [10]. If $P$ and $Q$ are two disjoint ordered sets, an ordered set $V$ is called an extension of $P$ by $Q$ if there is an ideal $P'$ of $V$ such that $P'$ is isomorphic to $P$ and the complement $V \setminus P'$ of $P'$ to $V$ is isomorphic to $Q$. The ideal extension problem for ordered sets is as follows: given two disjoint ordered sets $P$ and $Q$, construct (all) ordered sets $V$ that are (ideal) extensions of $P$ by $Q$. We are often interested in building more complicated semigroups, lattices, ordered sets, ordered or topological semigroups by using objects of a “simpler” structure, and this can sometimes be achieved by constructing ideal extensions. Equivalent extensions of ordered sets were considered by Kehayopulu and Shum in [11]. In the present paper, we introduce the concept of retract extension of ordered set and characterize retract extensions. As illustrative examples, we give an example of a retract extension and an example that is not a retract extension. In fact, Example 1 considered in [10] is an extension of an ordered set $P$ by an ordered set $Q$ ($P$ and $Q$ are disjoint), which is not a retract extension. The Hasse diagram of that extension was given in [10].

Let $(V, \leq)$ be an ordered set. A nonempty subset $P'$ of $V$ is called an ideal of $V$ if $a \in P'$ and $V \ni b \leq a$ implies that $b \in P'$ (see [1]). Each nonempty subset $P'$ of an ordered set $(V, \leq_V)$ with a relation “$\leq_{P'}$” on $P'$ defined by $\leq_{P'} := \leq_V \cap (P' \times P')$ is an ordered set. In the sequel, each subset $P'$ of an ordered set $(V, \leq_V)$ is regarded as an ordered set endowed with the order $\leq_{P'} := \leq_V \cap (P' \times P')$. We denote by $V \setminus P'$ the complement of $P'$ in $V$.

**Definition.** If $(P, \leq_P)$ and $(Q, \leq_Q)$ are two disjoint ordered sets, an ordered set $(V, \leq_V)$ is called an ideal extension (or just an extension) of $P$ by $Q$ if there exists an ideal $P'$ of $V$ such that

$$(P', \leq_{P'}) \cong (P, \leq_P) \quad \text{and} \quad (V \setminus P', \leq_{V \setminus P'}) \cong (Q, \leq_Q),$$

where $\leq_{P'} := \leq_V \cap (P' \times P')$ and $\leq_{V \setminus P'} := \leq_V \cap ((V \setminus P') \times (V \setminus P'))$ (see [10]).

Throughout the paper we use the following notation.

**Notation 1.** If $(V, \leq_V)$ is an extension of $P$ by $Q$, unless otherwise stated, we always denote by $\varphi$ and $\psi$ the isomorphisms

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Theorem 1. \( \varphi : (P, \leq_P) \to (P', \leq_{P'} \cap (P' \times P')) \) and \( \psi : (Q, \leq_Q) \to (V \setminus P', \leq_V \cap ((V \setminus P') \times (V \setminus P')) \), respectively.

An extension \( V \) of \( P \) by \( Q \) is also denoted by

\[ V(P, Q; \varphi : P \to P', \psi : Q \to V \setminus P'). \]

We denote by \( i_P \) the identity mapping on \( P \).

**Notation 2.** For every \( r \subseteq P \times Q \), we always denote by \( \bar{r} \) the set

\[ \bar{r} := \{(a, b) \in P \times Q \mid \exists (a', b') \in r \text{ such that } a \leq_P a', b' \leq_Q b}\].

Clearly, \( r \subseteq \bar{r} \).

The theorem below is the main theorem on ideal extensions of ordered sets given in [10].

**Theorem** (see [10; the theorem]). Let \( (P, \leq_P) \) and \( (Q, \leq_Q) \) be ordered sets such that \( P \cap Q = \emptyset \). Let \( r \subseteq P \times Q \) and \( V := P \cup Q \). We define a relation \( \leq_V \) on \( V \) as follows: \( \leq_V := \leq_P \cup \leq_Q \). Then \( (V, \leq_V) \) is an ordered set, \( P \) is an ideal of \( V \), and

\[ V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P') \]

is an extension of \( P \) by \( Q \).

Conversely, let \( (V, \leq_V) \) be an extension of \( P \) by \( Q \). Suppose there exists an \( r \subseteq P \times Q \) such that for the set \( \bar{r} \) defined above, we have

\[ \bar{r} = \{(a, b) \in P \times Q \mid \varphi(a) \leq_V \psi(b)\} \]

Then, the set \( P \cup Q \) endowed with the relation \( \leq_V \) mentioned in the first part of the theorem is an ordered set and \( (P \cup Q, \leq) = (V, \leq_V) \).

In the sequel, we consider extensions \( V(P, Q; \varphi : P \to P', \psi : Q \to V \setminus P') \) of \( P \) by \( Q \) for which there is an \( r \subseteq P \times Q \) such that for the set \( \bar{r} \) defined in Notation 2, we have

\[ \bar{r} = \{(a, b) \in P \times Q \mid \varphi(a) \leq_V \psi(b)\} \]

Such extensions are retract extensions (also equivalent extensions), denoted by \( V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P', r) \).

**Remark** (see [10; Proposition 1]) If \( V(P, Q; \varphi : P \to P', \psi : Q \to V \setminus P', r, \leq_V) \) is an extension of \( P \) by \( Q \) and \( r := \{(a, b) \in P \times Q \mid \varphi(a) \leq_V \psi(b)\} \), then \( \bar{r} = r \).

2. The main result

**Definition 1.** An extension \( V(P, Q; \varphi : P \to P', \psi : Q \to V \setminus P', r) \) of \( P \) by \( Q \) is called a retract extension if there is an isotope mapping

\[ \eta : Q \to P \text{ such that } (a, b) \in r \text{ implies } a \leq_P \eta(b). \]

**Theorem 1.** An extension \( V(P, Q; \varphi : P \to P', \psi : Q \to V \setminus P', r) \) of \( P \) by \( Q \) is a retract extension if and only if there is an isotope mapping

\[ g : V \to P \text{ such that } g(x) = \varphi^{-1}(x) \text{ for every } x \in P'. \]

**Proof.** \( \Rightarrow \). Let \( \eta : Q \to P \) be an isotope mapping such that \( (a, b) \in r \Rightarrow a \leq_P \eta(b) \). We consider the mapping

\[ g : V \to P | a \to \begin{cases} \varphi^{-1}(a) & \text{if } a \in P', \\ \eta(\psi^{-1}(a)) & \text{if } a \in V \setminus P'. \end{cases} \]

(1) The mapping \( g \) is isotope.

Let \( a, b \in V \), \( a \leq_V b \). Then \( g(a) \leq_P (b) \).