ON AN EXPONENTIAL SUM

P. Ding

Let \( p \) be a prime number, \( n \) be a positive integer, and \( f(x) = ax^k + bx \). We put

\[
S(f, p^n) = \sum_{x=1}^{p^n} e \left( \frac{f(x)}{p^n} \right),
\]

where \( e(t) = \exp(2\pi it) \). This special exponential sum has been widely studied in connection with Waring’s problem. We write \( n \) in the form \( n = Qk + r \), where \( 0 \leq r \leq k - 1 \) and \( Q \geq 0 \). Let \( \alpha = \text{ord}_p(k) \), \( \beta = \text{ord}_p(k - 1) \), and \( \theta = \text{ord}_p(b) \). We define

\[
Q = \begin{cases} \frac{\theta + n}{\theta + k} & \text{if } \theta \geq \alpha, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( J = \left\lbrack \frac{\alpha}{\theta} \right\rbrack \). Moreover, we denote \( V = \min(Q, J) \).

Improving the preceding result, we establish the theorem.

**Theorem.** Let \( k \geq 2 \) and \( n \geq 2 \). If \( p > 2 \), then

\[
|S(f, p^n)| \leq \begin{cases} \frac{p^{1-V}}{p} & \text{if } n \equiv 1 \pmod{k}, \\ (k-1, p-1)p^{V} & \text{if } n \equiv 0 \pmod{k}. \end{cases}
\]

An example showing that this result is best possible is given. Bibliography: 15 titles.

**1. Introduction.** Let \( p \) be a prime, \( n \) a positive integer, and \( f(x) = ax^k + bx \). In this paper we are concerned with the exponential sum

\[
S(f, p^n) = \sum_{x=1}^{p^n} e \left( \frac{f(x)}{p^n} \right),
\]

where \( e(t) = \exp(2\pi it) \).

Pioneered by the work of Hardy and Littlewood, this special case has widely been studied in connection with Waring’s Problem. Davenport and Heilbronn [5] showed that

\[
S(f, q) \ll q^{\eta + \varepsilon} (q, b) \prod_{(a, q) = 1},
\]

where \( \eta = 2/3 \) if \( k = 3 \) and \( \eta = 3/4 \) if \( k \geq 4 \). A well-known work of Weil [13] implies that

\[
S(f, p) \leq (k-1)p^{1/2} \text{ if } p \nmid a.
\]

By making use of (1.3), Hua [7] was able to reduce \( \eta \) to \( 1/2 \) for all \( k \geq 2 \).

Improving the work of Loxton and Smith [9], Loxton and Vaughan [10] showed that

\[
|S(f, p^n)| \leq (k-1)p^{\frac{n-\varepsilon}{2}} (D, p^n)^{1/2},
\]

where \( D \) is the different of \( f(x) \) and \( \tau = 1 \) or 0 according to \( p \leq k \) or \( p > k \).

Let \( \text{ord}_p(Y) \) denote the normalized exponent valuation on the \( p \)-adic field. For any positive integer \( Y \), \( \text{ord}_p(Y) \) is the largest integer \( v \) satisfying the condition \( p^v | Y \). For convenience, we write \( \theta = \text{ord}_p(b), \alpha = \text{ord}_p(k), \beta = \text{ord}_p(k - 1), \) \( k = p^\alpha k_1, b = p^\beta b_1, \) and \( k - 1 = p^\beta k_0 \). Then \( (p, k_1) = (p, b_1) = (p, k_0) = 1 \).
Recently, Dabrowski and Fisher [4] have established the following estimates under the conditions \((p,b) = (p,k) = 1\). If \(p > 2\), then

\[
|S(f, p^n)| \leq \begin{cases} 
(k-1)p^{1/2} & \text{if } \beta = 0, \\
(k-1)p^{-\beta/2}p^{n/2} & \text{if } \beta \geq 1 \text{ and } n \geq 3\beta+2, \\
(k-1, p-1)^{\min(\beta, \frac{n}{2})}p^{n/2} & \text{if } n \text{ is even}, \\
(k-1, p-1)^{\min(\beta, \frac{n}{2}-1)}p^{(n+1)/2} & \text{if } n \text{ is odd}.
\end{cases} 
\] (1.5)

Similar results hold for \(p = 2\).

For relatively large \(k\), Ye [15] modified (1.5) for \(\beta = 0\) and for \(1 \leq \beta \leq (k-2)/3\) by using \((\varphi(p^n) - k + 1)\) instead of \((k-1)\). This supersedes (1.5) when \(k\) is close to \(\varphi(p^n)\) for \(\beta = 0\) and for \(1 \leq \beta \leq (k-2)/3\), specifically, when \(k\) lies in the interval \((\frac{1}{2}\varphi(p^n) + 1, \varphi(p^n))\).

Cochrane and Zheng [3] proved the following general result.

Let \(k \geq 2\), \(n \geq 2\), and \(\gamma = \text{ord}_{p}(a) \leq n-2\). Then for odd prime \(p\),

\[
|S(f, p^n)| \leq (k-1, p-1)^{\min(\beta, \frac{n}{2})}p^{\frac{1}{2}}(\varphi(p^n))^{\frac{1}{2}},
\] (1.6)

and for \(p = 2\),

\[
|S(f, 2^n)| \leq 2 \cdot 2^{\frac{1}{2}}(\varphi(p^n))^{\frac{1}{2}}(2^n)^{\frac{1}{2}}.
\] (1.7)

We write \(n = Qk + r\) with \(0 \leq r \leq k - 1\) and \(Q \geq 1\). Let

\[
\zeta = \begin{cases} 
\frac{\theta - \alpha}{k-1} & \text{if } \theta \geq \alpha, \\
0 & \text{otherwise},
\end{cases}
\]

and \(J = [\zeta]\). Moreover, we write \(V = \min(Q, J)\). Define

\[
\theta_j = \theta + j - \lambda \quad \text{for} \quad j = 0, 1, 2, \ldots, V.
\] (1.8)

Let \(t = \text{ord}_{p}(f'(x))\), \(\mu\) be a root of the underlying congruence

\[
p^{-t}f'(x) \equiv 0 \pmod{p}
\] (1.9)

with multiplicity \(m_\mu\), and \(A\) be the set of roots of (1.8). Write \(M = \max_{\mu \in A} m_\mu\) and \(m = \sum_{\mu \in A} m_\mu\).

Further define \(\sigma_\mu = \text{ord}_{p}(f(px + \mu) - f(\mu))\). For convenience, we write \(d = (k-1, p-1)\). We define a sequence of polynomials \(f_j\) as follows. Let \(f_0 = f\). For \(j = 1, \ldots, V\), let \(f_j = ax^k + px^{b_1}x\). For each \(f_j\), \(j \geq 1\), we define \(t_j, \mu_j, m_{\mu_j}, A_j\), and \(\sigma_{\mu_j}\) corresponding to \(t, \mu, m_\mu, A, \) and \(\sigma_\mu\), respectively.

The aim of this paper is to improve the above results by establishing the following theorem.

**Theorem.** Let \(k \geq 2\) and \(n \geq 2\). If \(p > 2\), then

\[
|S(f, p^n)| \leq \begin{cases} 
p^{\frac{1}{2}}(\varphi(p^n))^{\frac{1}{2}} & \text{if } n \equiv 1 \pmod{k}, \\
(k-1, p-1)^{\min(\beta, \frac{n}{2})}p^{(n+1)/2} & \text{if } n \not\equiv 1 \pmod{k}.
\end{cases}
\] (1.10)

The outcome of the theorem proves to be best possible, which is shown with an example in Sec. 4; some discussion is also given at the end of this paper.

### 2. Preliminary results

**Lemma 1** (see [3]). If \(p > 2\) and \(n \geq t + 2\), or \(p = 2\) and \(n \geq t + 3\), or \(p = 2\) and \(t = 0\), then if \(A = \phi\) then \(S(f, p^n) = 0\); if \(A\) is not empty, we have

\[
S(f, p^n) = \sum_{\mu \in A} e\left(\frac{f(\mu)}{p^n}\right)p^{\sigma_\mu - 1}S(g_\mu, p^{n-\sigma_\mu}),
\]

where

\[
S(g_\mu, p^{n-\sigma_\mu}) = \begin{cases} 
p^{n-\sigma_\mu} & \text{if } n < \sigma_\mu, \\
\sum_{x \equiv \mu \pmod{p^{n-\sigma_\mu}}} e\left(\frac{g_\mu(x)}{p^{n-\sigma_\mu}}\right) & \text{if } n \geq \sigma_\mu.
\end{cases}
\]