SUBEXPONENTIAL DISTRIBUTION FUNCTIONS IN R^d

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1. Introduction

Let $F(x)$ denote a distribution function (d.f.) in $\mathbb{R}^d$ with $F(0^+) = 0$ and $F(x) < 1$, $\forall x \in \mathbb{R}^d$. Let $X, X_1, X_2, \ldots, X_n$ denote independent random vectors with the same d.f. $F(x)$. Independent of $X$, let $N$ denote an integer-valued random variable with probability distribution $P(N = n) = p_n$. Let $S(0) = 0$ and, for $n \geq 0$, let $S(n+1) = S(n) + X_{n+1}$.

The new random variable (r.v.) $S(N)$ has d.f.

$$W(x) = P(S(N) \leq x) = \sum_{n=0}^{\infty} p_n P(S(n) \leq x) = \sum_{n=0}^{\infty} p_n F^{*n}(x),$$

where $F^{*n}(x)$ denotes the $n$-fold convolution of $F(x)$ and where $F^{*0}(x)$ denotes the unit mass at 0. The d.f. $W(x)$ is called subordinate to $F(x)$ with subordinator $\{p_n\}$. As in the univariate case in the paper, we shall assume that $N$ satisfies condition (A): $N$ has a generating function $F(z) = E(z^N)$ that is analytic at $z = 1$.

In the present paper, we discuss the relation between the asymptotic behavior of $1 - F(x)$ and that of $1 - F^{*n}(x)$ and $1 - W(x)$. It turns out that, as in the univariate case, there are many cases in which $1 - F^{*n}(x)$ asymptotically behaves as $n(1 - F(x))$ and $1 - W(x)$ behaves as $E(N)(1 - F(x))$. To specify the precise kind of asymptotic behavior, we present a form of multivariate subexponentiality.

The paper is organized as follows. In Sec. 2, we briefly recall some basic properties and definitions concerning univariate subexponential d.f. In Sec. 3, we introduce and study multivariate subexponential d.f.'s. In Sec. 4, we present a form of multivariate subexponentiality.

Without further comment, in the paper, we shall assume that all random vectors $X, Y, Z$, etc. are positive and have infinite support, i.e., the d.f. satisfies $F(0^+) = 0$ and $F(x) < 1$, $\forall x \in \mathbb{R}^d$. We also use the notation $\overline{F}(x) = 1 - F(x)$, and for vectors $x$ and $a$, we set $x^a = \min(x_i)$ and $a \ast x = (a_1x_1, a_2x_2, \ldots, a_dx_d)$.

2. Univariate Subexponential Distributions

In the one-dimensional case, many papers have been devoted to the tail behavior of subordinated d.f.'s. In doing so, the class of subexponential d.f.'s (notation: $S$) plays an important role. Extending the class $S$, Chover et al. [6, 7], introduced the class $S(\gamma)$, where $\gamma \geq 0$. To define these classes, let $F(x)$ denote a d.f. in $\mathbb{R}$ such that $F(0^+) = 0$ and $F(x) < 1$, $\forall x \in \mathbb{R}$. Also, let $f(s) = E(e^{sX})$ denote the generating function of $X$ or $F(x)$.

The d.f. $F(x)$ belongs to the subexponential class $S$ (notation: $F \in S$) if it satisfies

$$\lim_{x \to \infty} \frac{\overline{F}(x)^{\gamma}}{\overline{F}(x)} = 2.$$

The class $S$ is large and contains all d.f.'s with a regularly varying tail. Below, we bring together some important classes of functions. In each case, $u(x)$ denotes a positive and measurable function. For a review of these classes, we refer to [5, 16, 38].

The class $RV(\alpha)$: $u(x)$ satisfies $u(x) = x^\alpha$, $\forall x > 0$.

The class $ORV$: $u(x)$ satisfies $\limsup u(x)/u(t) < \infty$, $\forall x > 0$.

The class $L$: $u(x)$ satisfies $u(x + t)/u(t) = 1$, $\forall x \in \mathbb{R}$.

It is well known that $RV \subset L \cap ORV$ and that $\overline{F}(x) \in L \cap ORV$ implies that $F \in S$. Conversely, $F \in S$ implies that $\overline{F}(x) \in L$. Further basic properties can be found, e.g., in [12–15, 40].

Extending the class $S$, we say that $F \in S(\gamma)$ with $\gamma \geq 0$ if it satisfies the following three properties:

(i) $f(-\gamma) < \infty$;

(ii) $\lim_{x \to \infty} \overline{F}(x - y)/\overline{F}(x) = e^{\gamma y}$, for all $y \in \mathbb{R}$, notation: $\overline{F}(x) \in L(\gamma)$;

(iii) $\lim_{x \to \infty} \frac{\overline{F}(x)^2}{\overline{F}(x)} = 2f(-\gamma)$.

For $\gamma = 0$, this definition can be simplified and one can prove that $S(0) = S$.

The following result illustrates the use of subexponential d.f.’s in studying subordination in $\mathbb{R}$. Recall condition (A) from the introduction.

**Lemma 1** (see [11]). Suppose that $N$ satisfies condition (A). The following relations are equivalent: as $x \to \infty$

1. $F \in S$;
2. $F^{\ast n}(x) \sim nF(x), \forall n \geq 2$;
3. $\overline{W}(x) \sim E(N)F(x)$.

Lemma 1 shows that for $F \in S$ we have $P(S(n) > x) \sim nP(X > x)$, i.e., the tail distribution of the partial sums $S(n) = \sum_{i=1}^{n} X_i$ behaves asymptotically as $n$ times the tail distribution of $X$. Alternatively, let $M(n) = \max(X_1, X_2, \ldots, X_n)$ denote the sequence of partial maxima. Clearly, $P(M(n) \leq x) = F^n(x)$. It follows from Lemma 1 that $F(x) \in S$ if and only if $P(S(n) > x) \sim P(M(n) > x)$.

For the class $S(\gamma)$, a similar result holds.

**Lemma 2** (see [8, 11]). Suppose that $P(z) = E(z^X)$ is analytic at $z = f(-\gamma)$ and that $\overline{F}(x) \in L(\gamma)$ with $\gamma \geq 0$.

The following relations are equivalent: as $x \to \infty$

1. $F \in S(\gamma)$;
2. $\overline{W}(x) \sim P'(f(-\gamma))\overline{F}(x)$.

### 3. Multivariate Subexponential Distributions

To generalize some of the one-dimensional results we introduce several types of multivariate subexponential behavior. In [10], Cline and Resnick also introduced and studied a form of multivariate subexponential distributions. Their approach is formulated in terms of vague convergence and uses point-process arguments. Our approach is different and is more direct.

**Definition 3.** $F \in S(\mathbb{R}^d)$ if and only if

$$\forall x > 0 \text{ with } x^0 < \infty, \quad \lim_{t \to \infty} \frac{F^{\ast 2}(tx)}{F(tx)} = 2.$$ 

Note that the definition automatically assumes subexponential marginals. In our definition, we compare $1 - F^{\ast 2}(x)$ and $1 - F(x)$ along lines of the form $tx$. Alternatively, we can also compare $1 - F^{\ast 2}(x)$ and $1 - F(x)$ along curves of the form $c(t) * x$, where $c(t) = (c_1(t), c_2(t), \ldots, c_d(t))$ and $c_i(t) \to \infty$ as $t \to \infty$.

**Definition 4.** $F \in S(\mathbb{R}^d, c)$ if and only if

$$\forall x > 0 \text{ with } x^0 < \infty, \quad \lim_{t \to \infty} \frac{F^{\ast 2}(c(t) * x)}{F(c(t) * x)} = 2.$$ 

Another alternative is to compare $1 - F^{\ast 2}(x)$ and $1 - F(x)$ along regions.

**Definition 5.** $F \in WS(\mathbb{R}^d)$ if and only if

$$\forall x > 0 \text{ with } x^0 < \infty, \quad \lim_{b \to \infty} \frac{F^{\ast 2}(b * x)}{F(b * x)} = 2.$$ 

Again, both definitions automatically assume subexponential marginals. Note that, in these definitions, we force the limit to be equal to 2. An extension of this follows later in Sec. 5.

**Remark 6.** In the three definitions, we assumed that there is convergence for all $x > 0$ with $x^0 < \infty$. We can also consider the case where the convergence holds for each marginal and for some fixed $x > 0$ or some fixed set $A$ of values $x > 0$. In this case, we shall use the notations $S(\mathbb{R}^d, loc(A)), S(\mathbb{R}^d, c, loc(A))$, and $WS(\mathbb{R}^d, loc(A))$. All results below remain valid with this restriction.

As in the 1-dimensional case, we define $L(\mathbb{R}^d)$ as the set of measurable and positive functions $u: \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{t \to \infty} u(tx - a)/u(tx) = 1, \forall x > 0$ and $\forall a \geq 0$. In the case where $u(x) = F(x)$, we shall assume that this limit relation holds $\forall x > 0$, with $x^0 < \infty$. Similarly, we define the classes $L(\mathbb{R}^d, c)$ and $WL(\mathbb{R}^d)$. As in Remark 6, we also can restrict the definitions to some fixed $x > 0$ or to some fixed set $A$ of values $x > 0$.

In our first result, we show that $F \in S(\mathbb{R}^d)$ implies that $\overline{F} \in L(\mathbb{R}^d)$. A similar result holds for the classes $S(\mathbb{R}^d, c)$ and $WS(\mathbb{R}^d)$.

**Theorem 7.** (i) If $F \in S(\mathbb{R}^d)$, then $\forall a \geq 0$ and $\forall x > 0$ with $x^0 < \infty$ we have $\lim_{t \to \infty} F(xt - a)/F(xt) = 1$, i.e., $\overline{F}(x) \in L(\mathbb{R}^d)$.