THE KREIN STRING AND CHARACTERISTIC FUNCTIONS OF NON-SELF-ADJOINT OPERATORS

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The operator generated by the Krein string is investigated in the framework of the extension theory of symmetric operators. A simple proof of the complete non-self-adjointness of the operator is proposed. The scattering function of the string is obtained with the help of the Derkach–Malamud formula for characteristic functions of almost solvable extensions. Bibliography: 16 titles.

1. INTRODUCTION

In this paper, we study the non-self-adjoint operator

\[ L_S = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix} \]  

(1)

generated by a string \( S \) in the Hilbert space \( \mathcal{H} = L^2 [0, l] \oplus L^2_m [0, l] \). An explicit definition of the operator \( L_S \) is given in the next section.

In [12], M. A. Nudelman proved that the operator \(-B_S := iL_S\) is a maximally accretive operator; the characteristic function of this operator was studied as a transfer function of a conservative system with continuous time for which \( B_S \) is a basic operator. More precisely, in [12] the Lax–Phillips semigroup corresponding to a nonhomogeneous string \( S \) has been constructed; one may treat the characteristic function of the operator \( B_S \) as the scattering matrix of the string \( S \). On the other hand, according to [13] (see also [1, 3]), there exists a canonical bijection between conservative scattering systems and unitary Lax–Phillips semigroups. This approach allows one to calculate coefficients of the conservative scattering system \( \lambda_{C_S} \) that describes small oscillations of a nonhomogeneous string. It turns out that the operator \( B_S \) coincides with the basic operator of the above-mentioned system. In the same paper [12], the simplicity of this system (or equivalently, the complete non-self-adjointness of the operator \( iB_S \)) has been established; in addition, a formula connecting the characteristic function of the operator \( B_S \) with the coefficient \( \Gamma(\cdot) \) of dynamical compliance of the string \( S \) has been derived.

Recall that an operator is called completely non-self-adjoint if this operator has no nontrivial reducing subspaces for which the restriction of the operator to the subspace is self-adjoint.

In the present paper, we investigate the operator \( L_S \) in the framework of the extension theory. More precisely, we consider a minimal symmetric operator \( L_0 \) generated by the string \( S \), compute the deficiency indices for this operator, construct a boundary triplet for the adjoint operator \( L_0^* \), and calculate the corresponding Weyl function \( M(\cdot) \). It turns out that in the case of a singular string \( S \), \( M(\cdot) \) is easily expressed via the coefficient of dynamical compliance of the string \( S \). Furthermore, we propose a simple proof of the fact that the symmetric operator \( L_0 \) is completely non-self-adjoint.

The operator \( L_S \) is an almost solvable extension of \( L_0 \). To compute the characteristic function of the former operator, we use the Derkach–Malamud formula for characteristic functions of almost solvable extensions (see [7, 8]). It turns out that, under a special choice of the boundary triplet for \( L_0^* \), the characteristic function of \( L_S \) coincides with the characteristic function calculated by Nudelman in [12].

We also describe extensions of the operator \( L_0 \) that have scalar characteristic functions. Moreover, the calculation of characteristic functions is reduced to a simple purely technical procedure.

Notation: \( \mathcal{H} \) and \( \mathcal{H} \) are Hilbert spaces; \([\mathcal{H} \rightarrow \mathcal{H}] \) is the set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{H} \); \([\mathcal{H}] := [\mathcal{H}, \mathcal{H}] \); \( \text{dom}(A) \) and \( \text{ran}(A) \) are the domain and range of the operator \( A \), respectively; \( \ker(A) = \{ f \in \mathcal{H} : Af = 0 \} \) is the kernel of \( A \); \( \rho(A) = \{ \lambda \in \mathbb{C} : \ker(A - \lambda) = \{0\} \) and \( \text{ran}(A - \lambda) = \mathcal{H} \} \) is the resolvent set of \( A \); \( \text{Im} A := (\text{adj} A^*)/2i \) is the imaginary part of the operator \( A \).

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Assume that a function $m(\cdot)$ is nondecreasing on the interval $[0, l]$ and $m(0) = 0$. We also assume that $x = 0$ and $x = l$ are points of growth and continuity of $m(\cdot)$, i.e., $0 = m(+0) < m(x) < m(l - 0) = m(l)$, $x \in (0, l)$. The sign “)” denotes a closing bracket if $L := l + m(l - 0) < \infty$ (the case of a regular string) and a closing parenthesis if $L = \infty$ (the case of a singular string).

The equality

$$g(x) = D_m f(x) := -i \frac{df(x)}{dm(x)}, \quad x \in [0, l],$$

means that

$$f(x) = f(0) + i \int_0^x g(t) dm(t), \quad x \in [0, l],$$

where the integral is understood in the Lebesgue–Stieltjes sense (in the case $m(x) = x$, we write $D_m = D$).

We denote by $W_m[0, l]$ the Hilbert space of square-integrable functions $f$, $f \in L^2_m[0, l]$, with respect to the measure $dm$ that are absolutely continuous with respect to the Lebesgue measure $dx$ and such that $Df \in L^2[0, l]$. The space $W_m[0, l]$ is equipped with the graph norm

$$\|f\|_{W_m}^2 = \int_0^l |f(t)|^2 dm(t) + \int_0^l |Df(t)|^2 dt.$$

The Hilbert space $W^m[0, l]$ consists of Lebesgue square-integrable functions $f$, $f \in L^2[0, l]$, that are absolutely continuous with respect to the Lebesgue-Stieltjes measure $dm$ and such that $D_m f \in L^2_m[0, l]$. The norm of a function $f \in W^m[0, l]$ is defined by

$$\|f\|_{W_m}^2 = \int_0^l |f(t)|^2 dt + \int_0^l |D_m f(t)|^2 dm(t).$$

One may treat the function $m(\cdot)$ as the distribution of masses which are supported by a string stretched over the interval $[0, l]$. If the string tension equals 1, we obtain the following equation of small oscillations for the loaded string:

$$\frac{\partial^2 w(x, t)}{\partial t^2} = \frac{\partial^2 w(x, t)}{\partial x^2} \frac{1}{\partial m(x)}, \quad x \in [0, l], \quad t \in [0, +\infty).$$  \hspace{1cm} (2)

Here $w$ is the displacement of the string from the equilibrium and $t$ denotes time.

Equation (2) generates the following operator (see, for example, [12]):

$$L_S = \left( \begin{array}{cc} 0 & -D_m \\ -D_m & 0 \end{array} \right);$$  \hspace{1cm} (3)

this operator acts in the Hilbert space $\mathcal{H} = L^2[0, l] \oplus L^2_m[0, l]$ (in addition, we assume that elements of the first component of the orthogonal sum are constant on mass-free intervals). The domain of the operator $L_S$ consists of pairs

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad f_1 \in W^m[0, l], \quad f_2 \in W_m[0, l],$$  \hspace{1cm} (4)

satisfying the following boundary condition:

$$f_1(0) = f_2(0).$$  \hspace{1cm} (5)

Moreover, in the case of a regular string we consider two variants of the right-hand side boundary condition:

$$f_1(l) = 0 \quad \text{or} \quad f_2(l) = 0.$$  \hspace{1cm} (6)

We treat the boundary condition (5) as the unit friction (concerning strings with friction, see [2, 10, 15]).