THE ADVANTAGE OF CAPITALISM VS. SOCIALISM DEPENDS ON THE CRITERION

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The talk given by the second author, L. A. Shepp, at the Linnik symposium, St. Petersburg, April 2005. We consider two distinct government tax policies towards companies: the republican policy gives tax breaks to the richer companies, while the democratic policy would perhaps give breaks to the weaker companies in hopes to keep them alive and so reduce unemployment. Which policy is better? We show that this depends on the optimization criterion, at least for the case of two companies, which is all that we can handle, in the stated mathematical formulation of the question.

Suppose that two standard Brownian motion processes are controllable by adding processes of nonnegative drifts with unit sum to each process nonanticipatively. Thus the controlled processes satisfy, for $i = 1, 2$,

$$dX_i(t) = \mu_i(t)dt + dW_i(t), \quad t \geq 0, \quad X_i(0) = x_i,$$

where $\mu_i(t) \geq 0$, $i = 1, 2$, $\mu_1(t) + \mu_2(t) \leq 1$, $t \geq 0$, and we are allowed to change the drifts $\mu_i$ instantaneously but without knowing the future increments of the independent standard Brownian motions $W_i$. It is assumed that if $X_i(t)$ hits zero for some finite $t$, then the company $i$ goes bankrupt and disappears.

We denote by $\nu$ the number, 0, 1, or 2, of $X_i$ that survive forever, i.e., that never hit zero. We show that to maximize $P(\nu = 2)$, the probability that $\nu = 2$, i.e., that both companies survive forever, the optimal policy is to always apply the drift to the bottom company. This is rigorously proven. We also argue that if the criterion is to maximize $E\nu$, the expected number of companies that survive forever, then the optimal policy sometimes pushes the upper company. We have no rigorous proof that this is correct, but the result has been verified numerically in two different ways and the evidence seems quite convincing. The problem is much harder if $n > 2$ or if the diffusion constants of the two Brownian motions are unequal.

The original mathematical problem was formulated by David Aldous.1

BACKGROUND AND METHODOLOGY INVOLVED

We begin with a review of stochastic optimization theory in order to explain some of the mathematical difficulties of the problem. Problems of stochastic optimization in finance and economics have a long history perhaps beginning with von Neumann and Morgenstern’s Theory of Games and Economic Behavior which motivated and initiated the theory of linear programming. Optimal control and optimal stopping problems came along later. Early examples are:

• Pricing the perpetual American option, posed by Samuelson, solved by McKeane;
• Maximizing the probability of getting to a specific wealth in a subfair casino, posed and solved by Dubins and Savage.

Such examples involved martingales and thus appeared to be a new problem direction, but Dynkin understood clearly that it reduces to linear programming. Though these problems were infinite-dimensional, the technique of smooth fit (which I learned in Russia 40 years ago) sometimes enabled optimal solutions to be guessed. Itô calculus played an important role in giving a simple criterion for the guessed optimal process to be a martingale or a submartingale. The technique was used in a number of cases to solve problems relevant to finance and economics, without having to solve the nonlinear differential equations which arise in the dynamic programming approach.

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1We consider also the more general version, suggested by Oded Palmon, where $\mu_i \geq -\delta$. In this version, money may be “stolen” from the rich and given to the poor.


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DIFFICULTY NUMBER ONE: SOME PROBLEMS ARE TOO DISCRETE

Suppose the mafia will kill you if you do not repay a debt to them. That is, you must turn a fortune $0 < f_0 < 1$ into one of size 1. Suppose 1 means one million rubles in a casino where there is only one subfair “double-or-nothing” bet. One can stake any amount $s \leq f_0$ and reach the fortune $f_1 = f_0 + s$ with probability $p < \frac{1}{2}$, or $f_1 = f_0 - s$ with probability $q = 1 - p > \frac{1}{2}$.

It was proved that bold play, where one stakes $s = \min(f_0, 1 - f_0)$ at each bet until one either goes broke or reaches the goal, is optimal.

Does it really need proof?

If, instead, after every bet an interest payment is extracted by dividing by $1 + a > 1$, so that $f_1 = \frac{f_0 + s}{1 + a}$ with probability $p < \frac{1}{2}$, or $f_1 = \frac{f_0 - s}{1 + a}$ with probability $q = 1 - p$, then it is even more “evident” that bold play is optimal, but this is in fact false! The optimal strategy appears to be very difficult to find.

The reason that this class of problems is difficult is that there is too much discreteness and Itô or differential calculus is not involved.

DIFFICULTY NUMBER TWO: SOME PROBLEMS ARE HIGHER DIMENSIONAL

An early one-dimensional problem (posed by Samuelson and solved by McKean in 1965), now “easy,” is to price the perpetual American option:

$$V(x) = \sup_{\tau} E_x (X(\tau) - k)^+ e^{-r\tau},$$

where

$$X = X(t) = x_0 e^{\sigma W(t) + (\frac{\sigma^2}{2} - \mu) t},$$

or

$$dX(t) = X(t)(\mu dt + \sigma dW(t)), \quad X(0) = x_0.$$  

Using Itô calculus and the principle of smooth fit, the value of the option was found and the optimal $\tau = \tau_a$ was shown to be the first passage time to a specific level $a > k$.

Xin Guo generalized this result to models involving arbitrage and insider information in her (2000) thesis.

For problems in economics rather than finance, one makes corporate decisions, and to study this, Radner introduced a model under which optimal decisions of direction and profit taking can be studied. In Radner’s model, the net worth of the company at time $t$ follows the SDE

$$dX(t) = \mu I(t) dt + \sigma I(t) dW(t) - dZ(t), \quad x(0) = x_0,$$

where $I(t) \in \{1, \ldots, n\}$ and $Z(t) \uparrow$ is the total dividends or profit taken up to time $t$. The set $S = \{(\mu_i, \sigma_i), i = 1, \ldots, n\}$ are the available corporate directions, and one seeks to maximize the total discounted expected profit

$$V(x_0, S) = \sup_{Z,A} E_{x_0} \int_0^{\tau_0} e^{-rt} dZ(t)$$

until bankruptcy at time $\tau_0 \leq \infty$ by choosing a nonanticipating corporate direction $I(t)$ and a nonanticipating profit-taking policy $Z(t)$. The value $V(x_0)$ is the true worth of the company, and the complete solution was found.

The company in Radner’s model goes bankrupt in a finite time with probability one.

This may be thought of as a consequence of unrestrained profit maximization.

The above problems that have been solved up to now in finance, economics, or engineering, have always involved a single Brownian motion. This effectively reduces the guess of the optimal control boundary to guessing a single threshold, possibly one for each value of the parameter, and smooth fit enables the threshold to be guessed, because the Itô equation is one-dimensional and it is only necessary to solve an ordinary differential equation to apply smooth fit. But the problem of this paper involves two independent Brownian motions, and hence its solution breaks new ground.