THE CENTRALIZER ALGEBRA OF THE DIAGONAL ACTION OF THE GROUP $GL_n(\mathbb{C})$ IN A MIXED TENSOR SPACE

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We consider the walled Brauer algebra $Br_{k,l}(n)$ introduced by V. Turaev and K. Koike. We prove that it is a subalgebra of the Brauer algebra and that it is isomorphic, for sufficiently large $n \in \mathbb{N}$, to the centralizer algebra of the diagonal action of the group $GL_n(\mathbb{C})$ in a mixed tensor space. We also give the presentation of the algebra $Br_{k,l}(n)$ by generators and relations. For a generic value of the parameter, the algebra is semisimple, and in this case we describe the Bratteli diagram for this family of algebras and give realizations for the irreducible representations. We also give a new, more natural proof of the formulas for the characters of the walled Brauer algebras. Bibliography: 29 titles.

INTRODUCTION

In this paper, we study two families of embedded finite-dimensional algebras that appeared in the theory of classical groups and in modern studies of quantum groups: the Brauer algebras and their important subalgebras, the walled Brauer algebras.

The history of appearance of these algebras is as follows. Assume that a classical group $(GL_n(\mathbb{C}), O_n(\mathbb{C}), Sp_n(\mathbb{C}))$ acts in a finite-dimensional vector space $V$. Consider the diagonal action of this group in the tensor product $V^\otimes f$ defined by the formula

$$A \cdot (v_1 \otimes \ldots \otimes v_f) = Av_1 \otimes \ldots \otimes Av_f.$$

Also consider the action of the symmetric group $S_f$ in this space:

$$\sigma \cdot (v_1 \otimes \ldots \otimes v_f) = (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(f)}), \quad \sigma \in S_f.$$

In the theory of Brauer algebras, the letter $n$ traditionally denotes the parameter of the algebra (the dimension of the space $V$), and the number of tensor factors is denoted by $f$, so that the notation for the symmetric group is quite unusual: $S_f$. It is not difficult to check that the constructed actions of the groups $GL_n(\mathbb{C})$ and $S_f$ commute with each other. In fact, a much stronger assertion holds; namely, as I. Schur proved in his thesis [26] in 1901, these actions generate the commutants of each other. This fact is known as the Schur–Weyl duality: it is one of the key points of the representation theory of both groups.

For example, the problem of decomposing the diagonal action of the full linear group in $\text{End}(V^\otimes f)$ into irreducible components is reduced to describing the commutant $C_f(GL_n(\mathbb{C}))$ of the image of this action [1]. Correspondingly, the same problem for the orthogonal group leads to considering the algebra $C_f(O_n(\mathbb{C}))$. However, in H. Weyl’s words, the latter algebra is “somewhat enigmatic,” which forced him to resort to other methods when studying the action of the orthogonal group.

In order to study the commutant $C_f(O_n(\mathbb{C}))$, R. Brauer [7] in 1937 introduced an associative algebra of diagrams $Br_f(n)$ and a homomorphism

$$br: Br_f(n) \rightarrow C_f(O_n(\mathbb{C})).$$

The definition of the algebra $Br_f(n)$ (the Brauer algebra) makes sense for every $n \in \mathbb{C}$; for $n \in \mathbb{Z}$ with sufficiently large absolute value, the algebra is not semisimple. For a comparatively long time, one managed neither to prove that the algebra is semisimple for $n \notin \mathbb{Z}$, nor to construct a description of the quotient by the radical for $n \in \mathbb{Z}$. Partial results were obtained in 1956 by W. Brown [8, 9]; however, a complete description of representations and branching for the family of algebras $\{Br_f(n)\}$ were obtained only in 1988 by H. Wenzl [29] with the help of conditional expectations and V. Jones’ basic construction.

The discovery by V. Jones and L. Kauffman of polynomial knot invariants (see [12, 13]), as well as investigations of quantum groups, allowed one to generalize these algebras. In 1989, J. Birman and H. Wenzl constructed a
two-parameter algebra $C_{n}(l, m)$ (the Birman–Wenzl algebra), of which the Brauer algebra turned out to be a particular case (see [5, 21]). A generalization of the constructed invariants led V. G. Turaev [27] to introducing, in the same year, algebras $H_{k, l}(x, y)$ that describe invariants of links. For an appropriate choice of the parameters, $H_{k, l}(x(n), y(n)) = H_{k, l}(n) \subset B_{r+1}(n)$.

In connection with the paper [2] by A. M. Vershik and A. Yu. Okounkov (in its first version published in Selecta Math., see [23]), V. G. Turaev drew A. M. Vershik’s attention to the fact that perhaps one can study the algebras introduced by him with the methods suggested in [2]. At the time, it was not clear whether these algebras differ from the group algebras of the symmetric groups; one did not know their relation to the Brauer algebras. The problem posed by A. M. Vershik to the author was to study these questions.

The paper is organized as follows. In Secs. 1 and 2, we give the definitions of the Brauer algebra $B_{r}(n)$ and the algebra $H_{k, l}(n)$. In Sec. 3, we show that if we consider the action of the group $GL_{n}(\mathbb{C})$ in the space of mixed tensors $V^{\otimes k} \otimes V^{\otimes l}$ and denote by $C_{k, l}(GL_{n}(\mathbb{C}))$ the commutant of this action, then

$$br(H_{k, l}(n)) \cong C_{k, l}(GL_{n}(\mathbb{C})), \quad (1)$$

under the identification $V^{\otimes k} \otimes V^{\otimes l} \cong V^{\otimes k+l}$. (When preparing the paper for publication, we found out that the remarkable paper [26] by A. M. Vershik and A. Yu. Okounkov, see [23], introduces a homomorphism $br$ that is injective on $H_{k, l}(n)$ for $k + l = 2n$.)

Below, in Sec. 4, we give a presentation of the algebra $H_{k, l}(n)$ by generators and relations. In Sec. 5, we introduce Jones’ basic construction, which is needed in Sec. 6 to describe the irreducible representations and the branching graph for the family of algebras $\{H_{k, l}(n)\}$. (When preparing the paper for publication, we found out that the algebra $H_{k, l}(n)$ was introduced in [17, 6], with property (1) taken as a definition, and the results of this paper were partially proved in [6, 15, 19]; see also [11].) In Sec. 7, we construct a realization of representations of the algebra $H_{k, l}(n)$ that allows us to suggest, in Sec. 8, a new, more natural, method for calculating the characters of this algebra (a similar construction for the Brauer algebra itself, as well as the method for calculating the characters, were suggested by S. V. Kerov, see [14]).

The author is grateful to A. M. Vershik for setting the problem and constant attention to his work.

1. Definition of the Brauer algebra

We need the definition of the Brauer algebras $B_{2}$ and $B_{r}(n)$, see [7]. As a vector space, the $\mathbb{C}(x)$-algebra $B_{r}$ is generated by diagrams consisting of two rows of $f$ points, where each point is joined by an arc with exactly one other point. For example, a diagram for $B_{3}$ can look as follows:

Given diagrams $a, b \in B_{r}$, the product $a \cdot b$ is constructed as follows. Draw the diagram $b$ under the diagram $a$ so that the lower row of points for $b$ coincides with the upper row of points for $a$; then delete the middle row of points. This operation results in $d$ cycles and a graph $c$ without cycles. Then $a \cdot b = x^{d}c$. For example,

$$a = \begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array}, \\
\begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array} = b, \\
a \cdot b = \begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array} = x \begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array}.
$$

The $\mathbb{C}$-algebra $B_{r}(n)$, $n \in \mathbb{C}$, is generated, as a vector space, by the same basis of diagrams. The product of diagrams is constructed in the same way as for $B_{r}$, but the variable $x$ is specialized to $n$.

**Theorem 1.1** (Brauer [7]). For $n \in \mathbb{N}$, there exists a homomorphism $br : B_{r}(n) \rightarrow \text{End}(V^{\otimes f})$ of the Brauer algebra to the commutant $C_{f}(O_{n}(\mathbb{C}))$ of the orthogonal group $O_{n}(\mathbb{C})$.

**Remark 1.2** Already H. Weyl, in his monograph [1, Chap. V], observed that the homomorphism $br$ is injective for $n \geq 2f$. W. Brown [9] proved that it is injective if and only if $n \geq f$.

**Remark 1.3** The symmetric group $S_{n}$ can be isomorphically embedded into the Brauer algebra $B_{r}(n)$: a permutation $\sigma$ is associated with the diagram that joins the $i$th point of the upper row with the $\sigma(i)$th point of the lower row. We will identify $\sigma$ with the corresponding diagram and the whole group algebra $\mathbb{C}[S_{n}]$ with the corresponding subalgebra in $B_{r}(n)$. The mapping $br$ sends the subalgebra $\mathbb{C}[S_{n}]$ of the algebra $B_{r}(n)$ to the commutant of the diagonal action of $GL_{n}(\mathbb{C})$ in the space $V^{\otimes f}$ (see the Introduction).