EQUATIONS ON GRAPHS IN $\mathcal{H}\infty$-OPTIMAL CONTROL PROBLEMS

A. E. Barabanov

Abstract. The method of reduction of a control problem to a system of coupled differential equations is indicated, and the simplest particular case is considered.

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1. Introduction

The study of properties of systems of differential equation whose boundary conditions are coupled in a certain sequence forming a certain graph were initiated under the guidance of Yu. V. Pokornyi in later 1980s [4, 5]. The main properties of systems of Sturm–Liouville equations with boundary conditions at the nodes of such a graph were studied. The ordinary mechanical treatment of vibrating coupled bars is a natural application of the constructed conceptual mathematical theory, which generalizes the classical results. The study of optimal control systems [1–3] pertaining to the direction of the $\mathcal{H}\infty$-optimization conducted in recent years posed close problems of both theoretical character (the uniqueness of solutions) and the efficiency of computational schemes. In these works, a new method for the solution of the problem of minimax controller synthesis in the linear-quadratic game problem was proposed. This method is directly extended from ordinary linear control objects with constant coefficients to systems with infinite-dimensional state spaces, in particular, to the delayed equations. It turned out that the value of the game in the minimax control problem coincides with the maximal value of the level of $\gamma$ for which there exists a nonzero solution of a certain homogeneous system of equations with linear conditions at the interior points. The obtained conditions can have a cyclic character: the values at the endpoint of the delay interval are coupled with the values at its initial point, and the conditions in the interior points couple one-sided limits to the left and to the right. Despite the fact that the obtained system does not always belong to the Sturm–Liouville class, it can be considered as a generalization of the studied problems of differential equations on graphs. In this work, a method of reduction of a control problem to a system of coupled differential equations and the simplest partial case are considered.

Acknowledgment. This work was partially supported by the Russian Foundation for Basic Research (project No. 04-01-00084).


2. Statement of the Problem

Let a control object be described by a linear stationary equation

\[ a(p)y(t) = b(p)u(t) + c(p)v(t) \]

in which \( a(\cdot) \), \( b(\cdot) \), and \( c(\cdot) \) are the Fourier transforms of certain distributions on the semiaxis \([0, +\infty)\), \( p = \frac{d}{dt} \) is the differentiation operator, \( y \) is the output of the object, \( u \) is the control, and \( v \) is the perturbation. The initial data are assumed to be zero, i.e., \( y(t) = u(t) = v(t) = 0 \) for \( t < 0 \).

Let \( \gamma > 0 \). It is required to find a non-look-ahead controller that for any function \( v \in L^2(0, \infty) \) ensures the existence of the Lebesgue square integrable solution of the closed system, \( y, u \in L^2(0, \infty) \), and also the fulfillment of the following \( \gamma \)-contraction condition:

\[
\int_0^\infty F(y(t), u(t)) \, dt < \gamma^2 \int_0^\infty v(t)^2 \, dt \quad \text{for} \quad v \neq 0,
\]

where \( F(y, u) \) is a quadratic form with a given matrix \( F_0 \). Controllers ensuring the fulfillment of these properties are said to be \( \gamma \)-contracting. It is also required to give a parametric description of the set of all \( \gamma \)-contracting controllers.

This statement is known as the standard \( H^\infty \)-optimal control problem in the case of complete information. It is convenient to reformulate it in terms of the “behavioral approach” introduced by Willems [6]. Let \( R(z) = (a(z), -b(z), -c(z)) \) be a row composed of the Fourier transforms of distributions with supports in the semiaxis \([0, +\infty)\). Introduce the vector of “manifest” variables \( x = (y, u, v) \) and the matrix of the quadratic form \( Q = \text{diag}\{F_0, -\gamma^2\} \). In this notation, the equation of the control object, the equation of the controller, and the objective inequality of \( \gamma \)-contraction are written as follows:

\[
R(p)x(t) = 0, \quad C(p)x(t) = 0, \quad \int_0^\infty x(t)^TQx(t) \, dt < 0 \quad \forall v \in L^2(0, \infty), \quad v \neq 0,
\]

where \( C(z) \) is the transfer function of the \( \gamma \)-contracting controller which is to be determined.

3. Algebraic Operator Method

The most popular means for the solution of the standard \( H^\infty \)-optimal control problem is the method of Riccati equations which, however, is efficiently applied only for stationary objects without delays. In delayed systems, the state space becomes infinite-dimensional and the solution of the Riccati equation becomes an integro-differential operator the search for which is a rather difficult computational problem. The spectral method competing with the Riccati equations in the computational respect also reduces to the solution of operator equations. In [2, 3], a new approach to the synthesis of optimal controllers applicable to integro-differential equations of the control object was proposed. This method also possesses the following advantages: the solution is conducted directly in terms of transfer functions of the open object, i.e., in the original engineering terms; to determine the class of all \( \gamma \)-contracting controllers, it suffices to solve only one system of linear equations with respect to the unknown transfer function.

Let \( \mathcal{R} \) be the set of Fourier transforms of all distributions on the real axis. It contains standard subsets: non-look-ahead subset \( \mathcal{R}_- \) and look-ahead subset \( \mathcal{R}_+ \). The set \( \mathcal{R}_- \) consists of Fourier transforms of distributions with supports in \([0, +\infty)\), and the set \( \mathcal{R}_+ \), with supports in \((-\infty, 0] \), respectively. Each of the sets \( \mathcal{R}_- \) and \( \mathcal{R}_+ \) is closed with respect to linear operations and multiplication.

Projectors of \( \mathcal{R} \) on \( \mathcal{R}_+ \) and on \( \mathcal{R}_- \) are denoted by \([\cdot]_+ \) and \([\cdot]_- \), respectively. Moreover, for any function \( f \in \mathcal{R} \), the function \( f - [f]_+ = [f]_0+ \) belongs to the set \( \mathcal{R}_- \), and the function \( f - [f]_- = [f]_0- \) belongs to the set \( \mathcal{R}_+ \). From here, in particular, it follows that the function \( f - [f]_+ - [f]_- = [f]_0 \) belongs to the set \( \mathcal{R}_- \cap \mathcal{R}_+ \) and, therefore, is a polynomial.