CONCIRCULAR VECTOR FIELDS ON SEMI-RIEMANNIAN SPACES

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Abstract. In this paper, we construct an analogue of concircular fields for semi-Riemannian spaces (i.e., for manifolds with degenerate metrics). We find a tensor criterion of spaces admitting the maximal number of concircular fields or having no such fields. We detect a gap in the distribution of dimensions of the space of concircular fields, which, in contrast to the corresponding gap in the case of pseudo-Riemannian manifolds, is lesser by 1. We also study some special types of concircular fields having no analogues for pseudo-Riemannian manifolds. The canonical form of the metric for some classes of semi-Riemannian spaces admitting concircular fields is obtained.

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Introduction

In 1939, Fialkow [3] introduced a vector field $\Phi$ on a pseudo-Riemannian manifold $(M, g)$ satisfying the condition

$$(\nabla_X \Phi) = \rho \cdot X,$$

where $X$ is an arbitrary differentiable vector field on $M$, $\rho$ is a scalar function, and $\nabla$ is the Levi-Civita connection. Such fields appear in the study of concircular mappings (i.e., conformal mappings preserving geodesic circles) and are called concircular fields (see [16]). In the literature, such fields are also called geodesic fields [9] (since integral curves of these fields are geodesic lines) and equidistant fields [11] (since they are gradient fields and normal congruences generated by them are equidistant). A particular case of concircular fields for $\rho = \text{const}$, the so-called converging fields of directions, was studied by P. A. Shirokov in [10]. Concircular fields play an important role in the theories of geodesic mappings and projective and conformal transformations. They were studied by a number of geometers: N. S. Sinyukov [11], H. L. Vries [15], A. V. Aminova [1], J. Mikeš [4], A. S. Solodovnikov [12], I. G. Shandra [5, 6], etc.

We specify a series of important properties of pseudo-Riemannian manifolds $(M, g)$ that possess a concircular field. If $\rho \neq 0$, then $(M, g)$ admits a nontrivial conformal transformation and a nontrivial geodesic mapping (and, therefore, the quadratic integral of geodesic lines). If several linearly independent concircular fields on the manifold exist, then all concircular fields are special, i.e., satisfy the condition

$$X(\rho) = K \cdot g(\Phi, X), \quad K = \text{const},$$

and the manifold admits a nontrivial projective transformation and a Killing vector field (the linear integral of geodesic lines). The space of concircular fields generates an ideal of the Jordan algebra of geodesic mappings [6]. Moreover, concircular fields also have interesting applications in physics [1, 13]. For example, in the de Sitter model, trajectories of time-like concircular fields determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis.

In this paper, we construct analogues of concircular fields for semi-Riemannian spaces (the so-called \(O(r, \mathbb{R}) \times GL(n - r, \mathbb{R})\)-structures). In Sec. 1, we recall necessary definitions and facts from the theory of pseudo-connections and semi-Riemannian spaces. Section 2 is devoted to the study of spaces either admitting the maximal number of concircular fields or having no such fields. We also detect a gap in the distribution of dimensions of the space of concircular fields, which, in contrast to the corresponding gap in the case of pseudo-Riemannian manifolds, is lesser by 1. In Sec. 3, we study some special types of concircular fields. In Sec. 4, we find the canonical forms of metrics and horizontal projectors for some classes of semi-Riemannian spaces admitting concircular fields. All investigations are performed locally in the class of sufficiently smooth function.

1. Preliminaries

1.1. Let \(M\) be a smooth \(n\)-dimensional manifold. We denote the ring of smooth functions on \(M\) by \(f(M)\), the Lie algebra of smooth vector fields on \(M\) by \(\mathfrak{X}(M)\), and arbitrary smooth vector fields on \(M\) by \(X, Y, Z, \) and \(W\).

**Definition 1.1.** A linear pseudo-connection on \(M\) is a pair of operators \((h; \nabla)\), where \(\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) and \(h\) is an affinor on \(M\) satisfying the following conditions [14]:

\[
\nabla_X(fY + Z) = f\nabla_XY + X(f) \cdot hY + \nabla_XZ, \quad (1.1a)
\]

\[
\nabla_{fX+Y}Z = f\nabla_XZ + \nabla_YZ, \quad X, Y \in \mathfrak{X}(M), \quad f \in f(M). \quad (1.1b)
\]

**Definition 1.2.** A linear quasi-connection on \(M\) is a pair of operators \((h; Q)\), where \(Q : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) and \(h\) is an affinor on \(M\) satisfying the following conditions:

\[
Q_X(fY + Z) = fQ_XY + hX(f) \cdot Y + Q_XZ, \quad (1.2a)
\]

\[
Q_{fX+Y}Z = fQ_XZ + Q_YZ, \quad X, Y \in \mathfrak{X}(M), \quad f \in f(M). \quad (1.2b)
\]

In the case where \(h = \text{id}\), any pseudo-connection (quasi-connection) is a linear connection on \(M\).

**Definition 1.3.** The tensors

\[
S(X, Y) = \nabla_XY - \nabla_YX - h[X, Y], \quad (1.3)
\]

\[
R(X, Y)Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X, Y]}Z, \quad (1.4)
\]

\[
\text{Ric}(X, Y) = \text{tr} R(X, Y) \quad (1.5)
\]

are called the **torsion tensor**, **curvature tensor**, and **Ricci tensor** of the pseudo-connection \((h; \nabla)\).

**Definition 1.4** (see [7, 8]). A linear pseudo-connection \((h; \nabla)\) is said to be **almost idempotent** if \(h^2 = h\). In this case, \(h\) is called the **horizontal projector** and \(v = \text{id} - h\) is called the **vertical projector**. An almost idempotent pseudo-connection is said to be **completely idempotent** if

\[
\nabla_XY = h\nabla_X(hY). \quad (1.6)
\]

A manifold on which a completely idempotent pseudo-connection \((h; \nabla)\), \(\text{rk} h = r\), is given is defined by \(A^r_n\).