THEORY OF SPECTRAL SEQUENCES. II

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ABSTRACT. In this paper, we continue to discuss the theory of spectral sequences in Abelian categories. In this connection, attention is paid to the manifestation of different dualities in the theory of spectral sequences. The duality in locally convex, topological vector spaces is of special interest. We use results of functional analysis for solving (co)homology problems (theorems on nondegenerate pairing (co)homology) of manifolds.

1. Introduction

Duality in mathematics is well known. However, for a great number of dual notions, constructions, theorems, theories, and approaches in investigation of those or other objects there is no rigorous definition of duality. Duality, with few exceptions (e.g., axioms of incidence in projective geometry), does not require axiomatization (most often axioms are dualized themselves) and lies in the nature itself of mathematical objects, as well as of mathematical logic.

Duality arises after first definitions in all mathematical (and not only) theories. For example, duality between objects and quotient objects in any category; in the definition of a morphism \( f : a \to b \) in a metacategory [26] with obvious dual objects: the domain \( a = \text{dom} f \) and the codomain \( b = \text{cod} f \) of a morphism \( f \), and, together with previous dualities, four mutual dual objects appear in the pre-Abelian category: the kernel \( \text{Ker} f \), the cokernel \( \text{Coker} f \), the image \( \text{Im} f \), and the coimage \( \text{Coim} f \) of a morphism \( f \). Morphisms as objects select dual classes: injections and surjections, which, together with the notion of weak equivalence (in closed model categories), describe some properties of objects, as well as arbitrary morphisms of this category. Thus, dual notions in (right) developed theory are multiplied and interlaced in a very improbable but regular way. Examples of such complicated and serious manifestations are duality of algebras and coalgebras [42], duality of (co)homology cooperations and operations in homotopy theory [42], dual Quillen and Sullivan theories, which in an equivalent way describe the rational homotopy type of a manifold via minimal models [31, 41], Poincaré–Verdier duality in sheaf theory [19], mirror dual Calabi–Yau manifolds in symplectic geometry [44], duality of the category of commutative \( \mathbb{Z} \)-algebras and the category of affine schemes, Cartier duality and the strange duality conjecture in algebraic geometry [10, 18, 30], duality in tensor categories and Langlands duality in quantum groups [15, 20].

In the theory of spectral sequences, in arbitrary Abelian categories [24] duality is presented in all of its known manifestations: filtrations and cofiltrations, monomorphisms and epimorphisms, subobjects and quotient objects, injective and projective exact couples, direct and inverse limits, (co)homology as subobjects and as quotient objects, different convergences to \( A^{-\infty} \) and to \( A^{\infty} \), right-half-plane and left-half-plane spectral sequences, and so on.

By the choice of additional axioms in the Abelian category for construction of spectral sequences, the first duality principle was exhibited as the so-called “inverting arrows” [6, 7, 10, 26] or by introducing a dual category whose objects are objects of the original category (and a bit more general, whose objects are bijective with objects of the original category) and whose morphisms between two objects are morphisms of the original category between these objects but in inverse order. This duality principle is mirror and we call it nominal, since, basically, it is nothing other than the renaming of notions, i.e., monomorphisms.
in the original category are called epimorphisms in the dual category, and vice versa, direct sums are called direct products, and vice versa, and so on. A question on the existence of a real category different from the considered dual category but equivalent to it, is of major interest. For example, the category of discrete Abelian groups is really dual to the category of compact Hausdorff groups and continuous homomorphisms given by the Pontryagin continuous character functor. Moreover, in Pontryagin duality there is an additional structure of pairing objects which is absent between objects in formally dual categories. Nevertheless, in the mentioned example the mirror principle is preserved.

The second duality principle is introduced in Abelian categories between subobjects and quotient objects of a fixed object [6]. This is a real duality, whose purpose is to describe in a different way the same object, e.g., (co)homology object, spectral sequence, and so on. Often it is also mirror but, as we shall see below, not always.

A connection between dual categories is contained in the form of duality isomorphism theorems (Poincaré duality, Alexander–Pontryagin duality, Lefschetz duality) or in the form of theorems about a link of dual objects (Pontryagin duality, Hahn–Banach theorem of an extension of a continuous functional which implies that a topological vector space and its topological dual space compose a dual pair). A deeper connection between dual categories is contained in theorems on adjoint functors. Its final connection is exhibited by introducing bonding objects (for example, (co)homology at infinity [37], binding and environment (co)homology [38], strong bonding (co)homology [23, 25]) and by examining relations between them. In our case, this is diagram (60) in [24] and here, below, diagrams (26) and (29), where, for example, in the second diagram in the Adams spectral sequence the stable homotopy group of the homotopy inverse limit of Adams filtration connects the target stable homotopy group of a spectrum (space) with the limit group of the projective exact couple to which this spectral sequence conditionally converges.

One of the purposes of this paper is to illustrate the third duality principle: in concrete categories, a symmetry between dual functors, objects, and constructions often disappears; it becomes apparent that one of them has been degenerated, which shows some inequality (disparity) between them. For example, in the category of $R$-modules, in particular, in the category of Abelian groups, the functor of direct limit is exact and at the same time the functor of inverse limit is not right-exact, in general. Every Hausdorff, locally convex, topological vector space is a projective limit of normed spaces, but only a Mackey space can be an inductive limit of normed spaces, or the inductive topology of a direct system of topological vector spaces defines the projective topology of a dual space, which is an inverse limit of dual spaces, but not vice versa.

2. Atiyah–Hirzebruch Spectral Sequence

Let $X$ be a CW-space or a CW-spectrum and $\{X_s\}$ be an increasing filtration on $X$ by its skeletons $X_s$ (more generally, CW-subcomplexes or CW-subspectra such that $\bigcup_s X_s = X$). Any spectrum $M$ gives rise to a generalized (extraordinary) cohomology theory $M^*(-)$ and homology theory $M_*(-)$, i.e., $M^m(X) = \{X, M\}_m = \{X, M\}_{-m}$ and $M_m(X) = \{S, M \wedge X\}_m$, where $\{Z, Y\}_m = \{\Sigma^m Z, Y\}$ is a group of stable homotopy classes, $S$ denotes the sphere spectrum, and $\Sigma^m$ is the $m$-suspension. For a space, one can consider reduced, as well as absolute theories; for spectra, theories are always reduced. Denote by $X = \circ$ the point spectrum, which we call a trivial spectrum. Note that $M^m(\circ) = M_m(\circ) = 0$, in contrast to the absolute cohomology and homology of a point, for which $M^m(*) = \{S^0, M\}_{-m} = \pi_{-m}(M)$ and $M_m(*) = \{S^0, M\}_m = \pi_m(M)$ are the coefficient groups.

Consider the injective and projective exact couples $(A^s, E^s)$ and $(C^s, E^s)$ with $A^s = M^m(X, X_{s-1})$, $C^s = M^m(X_s)$, and $E^s = M^m(X_s, X_{s-1})$, which define the Atiyah–Hirzebruch cohomology spectral sequence $\{E^r_s, d^r_s\}$ (with $E^s_{2,t} = H^s(X; \pi_{-t}(M))$, $s + t = m$, in the case of filtration by skeletons) and the exact injective and projective couples $(A^s, E^s)$ and $(C^s, E^s)$ with $A^s = M_m(X_{s-1})$, $C^s = M_m(X, X_{s-1})$, and $E^s = M_m(X_{s-1}, X_{s-2})$, which define the Atiyah–Hirzebruch homology spectral sequence $\{E^r_s, d^r_s\}$ (with $E^s_{2,t} = H_s(X; \pi_t(M))$, $s + t = m$, in the case of filtration by skeletons).

5531