SUBGROUPS OF $SL_n$ OVER A SEMILocal RING

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In the present paper, it is proved that if $R$ is a commutative semilocal ring all the residue fields of which contain at least $3n+2$ elements, then for every subgroup $H$ of the special linear group $SL(n, R)$, $n \geq 3$, containing the diagonal subgroup $SD(n, R)$ there exists a unique $D$-net $\sigma$ of ideals of $R$ such that $\Gamma(\sigma) \leq H \leq N_{\Gamma}(\sigma)$. In works by Z. I. Borewicz and the author, similar results were established for $GL_n$ over semilocal rings and for $SL_n$ over fields. Later I. Hamdan obtained a similar description for the very special case of uniserial rings. Bibliography: 76 titles.

Let $R$ be a commutative semilocal ring, $G = SL(n, R)$ be the special linear group of degree $n \geq 3$ over $R$, and $T = SD(n, R)$ be its subgroup of diagonal matrices. In the present paper, we describe subgroups of $G$ containing $T$ under some mild restrictions on the residue fields of $R$. More precisely, we prove the following result (we refer to Secs. 2 and 3 for an explanation of all definitions and notation used in this statement).

**Theorem 1.** Let $n \geq 3$ be a natural number and $R$ be a semilocal ring, all the residue fields of which contain at least $3n+2$ elements. Then the standard description of subgroups of $\Gamma = SL(n, R)$ containing the diagonal subgroup $T = SD(n, R)$ holds. In other words, for any such subgroup there exists a unique $D$-net of ideals $\sigma$ such that

$$\Gamma(\sigma) \leq H \leq N_{\Gamma}(\sigma).$$

The paper is organized as follows. In Sec. 1 we discuss previously known related results, and in Secs. 2 and 3 we introduce the necessary notation and state some lemmas used in the proof. In Sec. 4 we reduce the proof of Theorem 1 to the special case of subradical subgroups, which is embodied in Theorem 2, and in Secs. 5 and 6 we complete the proof of Theorem 2. Finally, in Sec. 7 we mention some further related problems.

1. Introduction

Since there are many dozens of papers of related interest, a brief historical survey is in order to put Theorem 1 in the context. We refer the reader to surveys [35–37, 39, 15, 16, 60, 73] for a much broader picture of the field and many further related references.

First we observe that for the general linear group, analogous results are substantially easier and known for some time. Subgroups of the general linear group $GL(n, R)$ over a semilocal ring $R$ containing the diagonal subgroup $D(n, R)$ were described in the 1970s in a series of papers by Zenon Borewicz and the present author (see [3, 4, 9, 10]). The ring $R$ was not even assumed to be commutative; in fact, most of the subtle aspects of the proof have been related to the noncommutativity of $R$! The methods of the above papers based on the use of pseudoreflections turned out to be extremely productive and served as a model for dozens of generalizations. In particular, the author and Elizaveta Dybkova carried them over also to other extended classical groups over rings such as the general symplectic group $GSp(2\ell, R)$ or the special orthogonal group $SO(n, R)$ (see [19, 71, 14, 15], and further references in [57]). Observe that the special orthogonal group $SO_n$ is already an extended Chevalley group, an analog of $GL_n$, while the true analog of the special linear group $SL_n$ is the spin group $Spin_n$. Lately Elizaveta Dybkova has obtained also very broad generalizations of these results to Bak’s unitary groups $GU(n, R, \Lambda)$ over a form ring [26–33]. In the case of skew-fields, the methods using pseudoreflections found definitive explanations in [75] and [63].

The case of the special linear group $SL_n$ turned out to be much harder even over fields. For finite fields $K$, $|K| \geq 13$, char $K \neq 2$, a description of subgroups containing a maximal torus, not even necessarily split, was obtained by Gary Seitz [66–68]. For infinite fields, there is a dramatic distinction between the cases $n \geq 3$ and $n = 2$: the group $SL_2$ behaves like a symplectic group rather than a linear one! A conceptual explanation of this phenomenon is offered in the work of Douglas Costa and Gordon Keller [55]; see also [22]. The case $n \geq 3$ was treated by the author, first in [11, I] for infinite fields and then in [11, II–IV] for all fields $K$ such that $|K| \geq 7$. While the proofs in [11, I] still relied on the presence of nontrivial “homology” in $SL(n, K)$ (i.e., a scalar multiple of a pseudoreflection $d \neq e$), they introduced calculations with conjugates of elements

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For all symplectic groups has been reduced to the case of \( \text{SL}_n \). In the sequel, these methods, based on calculations with long root elements, were successfully generalized to all Chevalley groups over infinite fields (see [16, 18, 74, 75, 21] and references therein).

For the group \( \text{SL}_n \), \( n \geq 3 \), itself, in the matrix language the basic trick consisted of considering the conjugate \( g \in \text{SL}_n \). Partial results were obtained by the present author and Dybkova in [19]. Namely, we could obtain a complete solution only when \( |K|/|K^2| > |K| \) or \(-1 \in K^2\); the latter case depended on a cute calculation by Oliver King [58]. That these conditions are in merit, and considerably simplify the analysis, can be seen from the fact that after 6 years precisely the same conditions appeared again in the proof by Li Shang Zhi [62] of a slightly weaker result on the maximality of the monomial subgroup in the special linear group. The final result was proved by Oliver King in [59] by a tremendous calculation, where a product of 32 factors from \( H \) and \( T \) has been employed to produce a nontrivial transvection in \( G \) in characteristic \( p \geq 7 \). King succeeded in finding complete proofs also in characteristics 2, 3, and 5, but they required separate arguments and/or calculations. Moreover, his proof works only starting with the field \( \mathbb{F}_2 \), while the result itself holds for all fields \( K \), \( |K| \geq 13 \), and for the field \( \mathbb{F}_2 \). Even today there is no unified proof that covers all cases!

As a byproduct of our study of \( \text{GL}_n \) over a local field [23], Ismail Hamdan [50, 51] succeeded in generalizing results of [11] to the group \( \text{SL}(n, R) \) over a commutative uniserial ring \( R \). Recall that a ring \( R \) is called uniserial if its ideals are linearly ordered by inclusion. Namely, Hamdan noticed that this condition allows one to apply the basic computational trick described above and to obtain a matrix with a zero entry in a subgroup of \( G \) containing \( T \) by a single commutation. The basic reason why it works is exceedingly simple: for uniserial rings, the condition \( \sum_{ij} g_{ij} g_{ij} = 0 \) implies that at least two of the products \( g_{ij} g_{ji} \) and \( g_{ii} g_{jj} \), \( r \neq s \), generate the same ideal and thus differ by an invertible factor. However, it required quite some work to take care for all details since now one cannot choose \( r \) and \( s \) arbitrarily. Observe that this result easily implies the standard description of overgroups of \( T \) for all \( R \) rings with large residue fields.

For some mysterious reason, at that time it has not been noticed that another easy modification of the approach of [11] gives the corresponding result for all commutative semilocal rings! In fact, the present paper hardly contains any new idea as compared with [4, 9, 11]; it only combines the calculations already present in those papers in a slightly different fashion. The proof of Theorem 1 works as follows. First we observe that, modulo the results for fields, the general case reduces to the study of subradical intermediate subgroups – this idea has already been presented in [9, 6]. Even for a subradical subgroup \( H \), we cannot guarantee that any of the equations \( g_{ij} = 0 \) has a nontrivial invertible root, for an arbitrary element \( g \in H \). However, we can do it if \( g \) itself is “homology” – this is exactly the only new observation, which we expound in Sec. 6. The present paper has been essentially written in 1998 in Bielefeld, see [76].

2. Basic notation

The following standard notation will be used throughout the paper. Let \( R \) be a commutative ring, \( R^+ \) be its multiplicative group, and \( J = J(R) \) be its Jacobson radical. Further, let \( G = \text{GL}(n, R) \) and \( \Gamma = \text{SL}(n, R) \) be the general linear and the special linear group of degree \( n \) over \( R \), respectively. For a matrix \( g \in G \), we denote by \( g_{ij} \) its entry in the position \((i, j)\), so that \( g = (g_{ij}) \), \( 1 \leq i, j \leq n \). As usual, \( g^{-1} = (g'_{ij}) \) denotes the inverse of \( g \). \( e \) denotes the identity matrix and \( e_{ij} \) is a standard matrix unit, i.e., the matrix whose entry in the position \((i, j)\) is 1 and all the remaining entries are zeros. Thus \( g = \sum g_{ij} e_{ij} \). As usual, we denote by \( \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \) the diagonal matrix with diagonal entries \( \varepsilon_1, \ldots, \varepsilon_n \).

By \( t_{ij}(\varepsilon) = \varepsilon + \varepsilon e_{ij} \) for \( \varepsilon \in R \) and \( 1 \leq i \neq j \leq n \), we denote an elementary transvection. If \( \varepsilon \in R^+ \) is invertible, we denote by \( d_{ij}(\varepsilon) = \varepsilon (\varepsilon - 1) e_{ij} \) an ‘elementary’ pseudoreflection. But the pseudoreflections do not belong to \( \Gamma \), and thus they must be replaced in calculations by the following two types of diagonal matrices with determinant 1. First, we consider products of two pseudoreflections

\[
d_{ij}(\varepsilon) = d_i(\varepsilon) d_j(\varepsilon^{-1}) = e + (\varepsilon - 1) e_{ii} + (\varepsilon^{-1} - 1) e_{jj}.
\]