MEAN VALUES CONNECTED WITH THE DEDEKIND ZETA FUNCTION

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UDC 511.466+517.863

For a cubic extension $\mathbb{K}_3/\mathbb{Q}$, which is not normal, new results on the behavior of mean values of the Dedekind zeta function of the field $\mathbb{K}_3$ in the critical strip are obtained.

Let $M(m)$ denote the number of integral ideals of the field $\mathbb{K}_3$ of norm $m$. For the sums

$$\sum_{m \leq s} M(m)^2$$

and

$$\sum_{m \leq s} M(m)^3$$

asymptotic formulas are derived. Previously, only upper bounds for these sums were known. Bibliography: 23 titles.

This is a preparatory section. In connection with the results presented here, see [1, 2]. Consider the group $S_3$. The elements of $S_3$ fall into the following three conjugacy classes: $C_1: (1);\ C_2: (1,2,3),(3,2,1);\ C_3: (1,2),(2,3),(3)$.

Hence there are three simple characters: the one-dimensional characters $\psi_1$ (the principal character) and $\psi_2$ (the other character determined by the subgroup $C_1 \cup C_2$), and the two-dimensional character $\psi_3$. $S_3$ is the Galois group of the non-Abelian extension $K_6$ of degree 6 of $\mathbb{Q}$, where $K_6$ is the normal closure of a cubic field $K_3$ over $\mathbb{Q}$ given by an irreducible polynomial $f(x) = x^3 + ax^2 + bx + c$ of discriminant $D$.

The fields $K_2 = \mathbb{Q}(\sqrt[3]{D})$ and $K_3$ are the intermediate extensions fixed under the subgroups $A_3$ and $\{(1),(1,2)\}$, respectively. The extensions $K_2/\mathbb{Q}, K_6/K_2$, and $K_6/K_3$ are Abelian. The Dedekind zeta functions (for the definition, see Sec. 2) satisfy the relations

$$\zeta_{K_6}(s) = L_{\psi_1}, L_{\psi_2}, L_{\psi_3}^2,$$

$$\zeta_{K_2}(s) = L_{\psi_1}, L_{\psi_2},$$

$$\zeta_{K_3}(s) = L_{\psi_1}, L_{\psi_3},$$

$$\zeta(s) = L_{\psi_1},$$

where

$$L_{\psi_2} = L(s, \psi_2, K_6/\mathbb{Q}) = L(s, \chi, K_2/\mathbb{Q}),$$

$$L_{\psi_3} = L(s, \psi_3, K_6/\mathbb{Q}) = L(s, \chi', K_6/K_2).$$

Here, the second column involves the Artin $L$-functions, and the third column involves the $L$-functions with Hecke characters (more exactly, $\chi(s) = (D/s)$).

Below, we assume that $K_2, K_3, K_6$ are the fields indicated above, and $D < 0$; $\varepsilon > 0$ is an arbitrary fixed number.

The function $L_{\psi_3}$ can also be interpreted in another way [3]. Let $\rho: S_3 \rightarrow GL_2(\mathbb{C})$ be the irreducible two-dimensional representation. Then $\rho$ gives rise to a cuspidal representation $\pi$ of $GL_2(\mathbb{Q})$. Let

$$L(s, \pi) = \sum_{n=1}^{\infty} a(n)n^{-s},$$

In particular, if $\rho$ is odd, i.e., $D < 0$, then $L(s, \pi) = L(s, F)$, where $F$ is a holomorphic primitive cusp form of weight 1 with respect to $\Gamma_0(|D|)$,

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad q = e^{2\pi iz}.$$

As usual, $L(s, \pi)$ denotes the $L$-function of the representation $\pi$; $L(s, F)$ denotes the Hecke $L$-function of the form $F$. Thus, $L_{\psi_3} = L(s, F)$,

$$\zeta_{K_6}(s) = \zeta(s)L(s, F).$$

Formula (1) will be used below.

This section improves known results connected with the behavior of the Dedekind zeta function of the cubic field $K_3$ in the critical strip. Let $K_n$ be an algebraic number field over $\mathbb{Q}$ of degree $n$. The Dedekind zeta function of $K_n$ is defined by the relation

$$\zeta_{K_n}(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s},$$

where the summation runs over all nonzero integral ideals in $K_n$. If $r_1$ is the number of real conjugates of $K_n$, $2r_2$ is the number of imaginary conjugates, and $\Delta$ is the discriminant of $K_n$, then the functional equation for $\zeta_{K_n}(s)$ may be written as

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \Gamma^{r_1} \left( \frac{1}{2} s \right) \Gamma^{r_2}(s) B^{-s} \zeta_{K_n}(s),$$

with

$$B = 2^{r_2} \pi^{r_1/2} (|\Delta|)^{-1/2}.$$

The Dedekind zeta function admits analytic continuation to the entire complex plane, and it only has a simple pole at $s = 1$ with residue

$$\text{res}_{s=1} \zeta_{K_n}(s) = \frac{2^{r_1} (2\pi)^{r_2} hR}{w \sqrt{|\Delta|}},$$

where $h$ is the class number of $K_n$, $R$ is the regulator of the field, and $w$ the number of roots of unity. Let $s = \sigma + it$. The critical strip for $\zeta_{K_n}(s)$ is the strip $0 \leq \sigma \leq 1$, and the critical line is the line $s = 1/2 + it$.

As was proved by Käufman [4, 5] and Heath–Brown [6],

$$\zeta_{K_n} \left( \frac{1}{2} + it \right) \ll t^{(n/6)+\varepsilon} \quad (t \geq 1).$$

Also the mean square

$$\int_1^T |\zeta_{K_n}(\sigma + it)|^2 \, dt, \quad 0 \leq \sigma \leq 1,$$

was estimated (see [7, 8]). For $K_3$ and the critical line, the upper bound obtained in [7] takes the form

$$\int_1^T |\zeta_{K_3} \left( \frac{1}{2} + it \right)|^2 \, dt \ll T^{3/2} \log^5 T.$$

Let $\sigma(K_n)$ be the lower bound of the numbers $\sigma$ such that

$$\int_1^T |\zeta_{K_n}(\sigma + it)|^2 \, dt \ll T^{1+\varepsilon}.$$

In [7, 8], it was shown that

$$\sigma(K_n) \leq 1 - \frac{1}{n}.$$

The theorem below improves the above results in the case of $K_3$.

**Theorem 1.** In the case of $K_3$,

(i) $\zeta_{K_3} \left( \frac{1}{2} + it \right) \ll t^{\frac{n/6}{r_2} + \varepsilon}$ \quad ($r \geq 1$);

(ii) $\int_1^T |\zeta_{K_3} \left( \frac{1}{2} + it \right)|^2 \, dt \ll T^{2+\varepsilon};$

(iii) $\sigma(K_3) \leq \frac{5}{8}$. 

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