DECAY OF SOLUTIONS OF THE FIRST MIXED PROBLEM FOR A HIGH-ORDER PARABOLIC EQUATION WITH MINOR TERMS

L. M. Kozhevnikova and F. Kh. Mukminov

UDC 517.956.4

FOR A HIGH-ORDER PARABOLIC EQUATION WITH MINOR TERMS


Abstract. In a cylindric domain $D = (0, \infty) \times \Omega$, where $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded domain, the first mixed problem for a high-order parabolic equation

$$
\begin{align*}
&u_t + (-1)^k D^k_x(a(x,y)D^k_x u) + \sum_{i=l}^{m} \sum_{|\alpha|=|\beta|=i} (-1)^i D^\alpha_y(b_{\alpha\beta}(x,y)D^\beta_y u) = 0, \\
&l \leq m, \quad k, l, m \in \mathbb{N}
\end{align*}
$$

is considered. The boundary values are homogeneous and the initial value is a finite function. In terms of the new geometrical characteristics of the domain, the upper estimate of the $L_2$-norm $\|u(t)\|$ of the solution to the problem is established. In particular, in domains $\{(x,y) \in \mathbb{R}^{n+1} \mid x > 0, |y_1 < x^n|, 0 < a < q/l, \}$, under the assumption that the upper and lower symbols of the operator $L$ are separated from zero, this estimate takes the form

$$
\|u(t)\| \leq M \exp(-x_kz^b)\|\varphi\|, \quad b = \frac{k-l-a}{k-l+a+2la}.
$$

This estimate is determined by minor terms of the equation. The sharpness of the estimate for a wide class of unbounded domains is proved in the case $k = l = m = 1$.

1. Introduction

Let $\Omega$ be an unbounded domain in

$$
\mathbb{R}^{n+1} = \{(x,y) \mid x \in \mathbb{R}, \ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_n\}
$$

located along the $Ox$-axis. We consider the first mixed problem for a parabolic equation:

$$
\begin{align*}
&u_t + (-1)^k D^k_x(a(x,y)D^k_x u) + \sum_{i=l}^{m} \sum_{|\alpha|=|\beta|=i} (-1)^i D^\alpha_y(b_{\alpha\beta}(x,y)D^\beta_y u) = 0, \\
&(t, x, y) \in D, \\
&l \leq m, \quad k, l, m \in \mathbb{N}, \\
&D^i_x u|_S = 0, \quad i < k, \\
&D^\alpha_y u|_S = 0, \quad |\alpha| < m, \\
&S = \{t > 0\} \times \partial\Omega
\end{align*}
$$

(1.1)

$$
\begin{align*}
&u(0, x, y) = \varphi(x,y), \\
&\varphi \in L_2(\Omega)
\end{align*}
$$

(1.2)

where $D$ is a cylindrical domain $D = \{t > 0\} \times \Omega$. Here $\alpha$ and $\beta$ are multi-indices, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The coefficients $b_{\alpha\beta}(x,y)$, $|\alpha| = |\beta| = m$, are measurable functions bounded for almost all $(x,y) \in \Omega$ and symmetric with respect to the multi-indexes $\alpha$ and $\beta$. The functions $b_{\alpha\beta}(x,y)$, $|\alpha| = |\beta| = m$, are uniformly continuous of $(x,y) \in \Omega$. We assume that the integral

$$
\sum_{i=l}^{m} \sum_{|\alpha|=|\beta|=i} \int_{\mathbb{R}^n} b_{\alpha\beta}(x,y)D^\alpha_y gD^\beta_y g \, dy \geq 0
$$

(1.3)

for all $g(x,y) \in C_0^\infty(\Omega)$ is nonnegative for almost all $x \in \mathbb{R}$. Moreover, we assume that the higher symbol is bounded from below for almost all $(x,y) \in \Omega$ and all $z \in \mathbb{R}_n$

$$
\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}(x,y)z^\alpha \bar{z}^\beta \geq \bar{b}|z|^{2m},
$$

(I)


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where \( \mathbf{z}^\alpha = z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_n^{\alpha_n} \). The function \( a(x,y) \) is assumed to be uniformly continuous for \( (x,y) \in \bar{\Omega} \) and to satisfy the inequality

\[
\hat{a} \leq a(x,y) \leq \bar{a}
\]  

with positive constants \( \hat{a} \) and \( \bar{a} \).

This work is devoted to the study of the large time behavior of the \( L_2(\Omega) \)-norm of the solution \( u(t,x,y) \) to the problem (1.1)–(1.3). Quite simple characteristics of an unbounded domain \( \Omega \) are suggested, in terms of which estimates of the decay rate as \( t \to \infty \) of the solution to problem (1.1)–(1.3) with the finite initial function \( \text{supp} \varphi \subseteq B(R_0,0,0) \) are obtained. Here and in what follows, we denote a ball of radius \( r \) and centered at \( (x,y) \) by \( B(r,x,y) \). The finiteness of the initial function is essential, since otherwise the rate of stabilization of the solution depends not only on the domain \( \Omega \), but on the initial function \( \varphi \) as well (see [9]).

The behavior of the \( L_2 \)-norm of the solution of mixed problems for second- and high-order parabolic equations as \( t \to \infty \) was studied by A. K. Gushchin [2], V. I. Ushakov, A. V. Lezhnev, F. Kh. Mukminov, A. F. Tedeev, I. M. Bikkulov, and others.

The equation

\[
u_t + (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta}(t,y)D^\beta u) = 0, \quad m \geq 1,
\]

was considered in [8]. The coefficients of this equation are measurable functions satisfying the inequalities

\[
c_1 \sum_{i=1}^n z_{m_i}^2 \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}z_\alpha z_\beta \leq c_2 \sum_{|\alpha|=m} z_\alpha^2
\]

for almost all \( t > 0, y \in Q \), and any set of numbers \( z_\alpha \), where \( c_1 \) and \( c_2 \) are positive constants, \( m_i = (0, \ldots, m_i, \ldots, 0) \) is a multi-index, whose \( i \)th component is equal to \( m \) and the rest are equal to zero. The domain \( Q \) was assumed to obey the following conditions: There exists a continuous nonincreasing function \( \mu(r), r > 0 \), satisfying the condition \( \lim_{r \to \infty} r^{1/2m}(r) = \infty \) and the inequalities

\[
0 < \mu(r) \leq \inf \left\{ \int_{Q^r} \left( \sum_{i=1}^n D^m_{y_i} g \right)^2 \, dy \right\} g(y) \in C^\infty_0(Q), \int_{Q^r} g^2 \, dy = 1 \}
\]

where \( Q^r = \{ y \in Q \mid |y_1| < r \} \). Under these conditions, for sufficiently large \( t \), an estimate

\[
\|u(t)\|_{L_2(Q)} \leq M \exp \left\{ -k \left( \int_{Q^r} g^2 \, dy \right)^{1/(2m-1)} \right\} \|\varphi\|_{L_2(Q)}
\]

was obtained for the solution of problem (1.1'), (1.2), (1.3) with a finite initial function \( \varphi \). Here \( t(r), t > 0 \), is the inverse function of \( F(r) = r/|\mu(r)|^{(2m-1)/2m} \).

Tedeev [10] considered a high-order quasi-linear parabolic equation of the form

\[
u_t + (-1)^m \sum_{|\alpha|=m} D^\alpha A_\alpha(t,y,u,Du,\ldots,D^mu) = 0, \quad m \geq 1.
\]

Here \( A_\alpha(t,y,\xi) \) are Caratheodory functions satisfying the conditions

\[
\sum_{|\alpha|=m} A_\alpha(t,y,\xi)\xi_\alpha^p \geq c_1 \sum_{|\alpha|=m} |\xi_\alpha^p|, \quad p \geq 2,
\]

\[
\sum_{|\alpha|=m} |A_\alpha(t,y,\xi)| \leq c_2 \sum_{|\alpha|=m} |\xi_\alpha^{p-1}|,
\]

for any vector \( \xi = (\xi^0, \xi^1, \ldots, \xi^m), \xi^i = (\xi^i_\alpha), |\alpha| = i \), where \( c_1 \) and \( c_2 \) are positive constants. The following conditions were assumed to be true for the domain \( Q \):

\[
\lim_{r \to \infty} \lambda(r) = 0, \quad \lim_{r \to \infty} r^2 \lambda(r) = \infty,
\]