EXTREMES AND STUDENT’S $t_2$-DISTRIBUTION

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Some new characterizations of Student’s distribution on two degrees of freedom by regresional properties of maximal order statistics are obtained. Bibliography: 5 titles.

The family of Student’s distributions is well known in the world of statisticians. We say that a random variable $X$ has $t_2$-distribution (Student’s distribution with $\nu$ degrees of freedom, where $\nu > 0$) if the density of distribution of $X$ has the form

$$f_\nu(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\pi \nu}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}, -\infty < x < \infty,$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$, $s > 0$, denotes the gamma-function. This family of distributions was introduced into statistical theory and practice at the beginning of the XXth century by William Gosset, who signed his mathematical papers by the pseudonym “Student.” At the beginning of the XXIst century, a series of papers [1−5] was published; the authors of these papers attracted the readers attention to some interesting properties of a representative of $t_2$-distributions with density

$$f_2(x) = (2 + x^2)^{-3/2}$$

and distribution function

$$F_2(x) = \frac{1}{2} \left(1 + \frac{x}{\sqrt{2 + x^2}}\right), -\infty < x < \infty.$$  \hspace{1cm} (3)

In particular, some characterizations of the $t_2$-distribution by properties of order statistics have been found. In this paper, we suggest two new characterizations for that type of distributions.

ORDER STATISTICS AND $t_2$-DISTRIBUTION

Let $X$, $X_1$, $X_2$,… be independent identically distributed random variables (rv’s) with a common distribution function $F(x)$, distribution density $f(x)$, and finite expectation. Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}, n = 1, 2, \ldots$, be the corresponding order statistics. We assume, in addition, that $f(x) > 0$ if $\gamma < x < \delta$, where $\gamma = \inf \{x : F(x) > 0\}$ and $\delta = \sup \{x : F(x) < 1\}$. It was shown in the paper [1] that the following statements are equivalent:

(a) $E(X_{1,3} + X_{2,3} | X_{2,3} = x) = 2x$ a.s.;

(b) $E(X_{1} + X_{2} + X_{3} | X_{2,3} = x) = 3x$ a.s.;

(c) $(F(x)(1 - F(x)))^{3/2} = cf(x)$, $\gamma < x < \delta$, $c > 0$;

(d) $\gamma = -\infty$, $\delta = \infty$, and $F(x) = F_2((x - \mu)/\sigma)$, $-\infty < \mu < \infty$, $\sigma > 0$,

where the distribution function $F_2$ is defined by (3).

The paper [1] was followed by the papers [2−5] in which new characterizations of the $t_2$-distribution were obtained and the result above was generalized to the case of samples of volume $n = 2k + 1$, $k = 1, 2, \ldots$. For example, in the paper [5], relations (4)−(7) were supplemented with the following two relations (which, as turned out, are equivalent to the above relations):

(e) $E(X_{1}X_{2}X_{3} | X_{2,3} = x) = cx$ a.s.;

(f) $E(X_{1,3}X_{3,3} | X_{2,3} = x) = c$ a.s.,

where $c$ is a negative constant. It was shown in [2] that, in particular, it is possible to characterize the $t_2$-distribution by the following relation:

$$E(X_1 + \ldots + X_{2k+1} | X_{k+1,2k+1} = x) = \frac{2k+1}{2m+1} E(X_1 + \ldots + X_{2m+1} | X_{m+1,2m+1} = x),$$

valid for some $k \neq m$.

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EXTREMES AND THE $t_2$-DISTRIBUTION

All the order statistics in equalitites (4) and (9) were constructed by the same sample $X_1, X_2, X_3$. Let us pass to a different situation. We replace the order statistics $X_{1,3}$, $X_{2,3}$ and $X_{3,3}$ which were used in the regression relations (4) and (9) by the following maxima:

$$M_1 = X_1 = X_{1,1}, \quad M_2 = \max\{X_1, X_2\} = X_{2,2},$$

and

$$M_3 = \max\{X_1, X_2, X_3\} = X_{3,3}.$$  

It turns out that it is possible to characterize the $t_2$-distribution by a relation that is similar to equality (2) and in which the order statistics $X_{1,3}$, $X_{2,3}$, and $X_{3,3}$ are replaced by the maximal order statistics $X_{1,1}$, $X_{2,2}$, and $X_{3,3}$. The following statement holds.

**Theorem 1.** Let $X_1$, $X_2$, and $X_3$ be independent, identically distributed random values with distribution function $F(x)$, density $f(x)$, and finite expectation. Assume that $f(x) > 0$ if $\gamma < x < \delta$, where $\gamma = \inf\{x : F(x) > 0\}$ and $\delta = \sup\{x : F(x) < 1\}$. Then the following two statements are equivalent:

(a) $E(M_1 + M_3|M_2 = x) = \frac{3}{2}x + a$ a.s., where $a$ is a constant;

(b) $\gamma = -\infty$, $\delta = \infty$, and $\tilde{F}(x) = F_2((x - 2a)/\sigma)$, $\sigma > 0$,

where the density function $F_2$ is defined by (3).

**Remark.** Let us note that the above characterization of the $t_2$-distribution has several peculiarities which distinguish it from the characterization of the same distribution by regression properties of the order statistics $X_{1,3}$, $X_{2,3}$, and $X_{3,3}$. First, the coefficients at $x$ in the right-hand sides of relations (4) and (11) are different. Second, while relation (4) holds for any representative of the family of $t_2$-distributions, i.e., this relation is preserved under shifts and changes of scale, relation (11) is preserved under changes of scale as well but responds to shifts.

Properties of the order statistics $X_{1,3}$, $X_{2,3}$, and $X_{3,3}$ and maxima $M_1$, $M_2$, and $M_3$ in their relation to the $t_2$-distribution can be united by the following characterization theorem.

**Theorem 2.** Assume that the conditions of Theorem 1 are satisfied. Assume, in addition, that the initial random values have zero expectation. Then the following two statements are equivalent:

$$E(M_1 + M_3|M_2 = x) = \frac{3}{4}E(X_{1,3} + X_{3,3}|X_{2,3} = x) \quad \text{a.s.;}$$

and

$$\gamma = -\infty, \quad \delta = \infty, \quad \text{and} \quad F(x) = F_2(x/\sigma), \quad \sigma > 0.$$  

**Proof of Theorem 1.** We apply the following quite obvious relations:

$$E(M_1|M_2 = x) = \frac{x}{2} + \int_{-\infty}^{x} u dF(u)/2F(x);$$

$$E(M_3|M_2 = x) = xF(x) + \int_{x}^{\infty} udF(u);$$

$$E(X_{1,3}|X_{2,3} = x) = \int_{-\infty}^{x} udF(u)/F(x);$$

and

$$E(X_{3,3}|X_{2,3} = x) = \int_{x}^{\infty} udF(u)/(1 - F(x)).$$

Assume that $EX = \mu$; denote

$$I(x) = \int_{-\infty}^{x} udF(u).$$

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