ON APPROXIMATION OF FUNCTIONS IN THE SPACES $L_p(\mathbb{R}^2)$ AND $L_p(\mathbb{R}^2_+)$ BY GENERALIZED SINGULAR INTEGRALS WITH POSITIVE KERNELS

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The following question is discussed: How the behavior of

$$\left\| \int_{\mathbb{R}^2} \Delta^r_{u,v}(f; \cdot) \varphi_n(u) \psi_m(v) dudv \right\|_{p, \mathbb{R}^2}$$

as $n, m \to \infty$, where $F \subset \mathbb{R}$,

$$\Delta^r_{u,v}(f; x, y) = \sum_{i, j=1}^{r} (-1)^{i+j} C^i_x C^j_y f(x + iu, y + jv) - f(x, y),$$

$\varphi_n$ and $\psi_m$ are positive kernels, is connected with the structural properties of $f$ that are characterized by the moduli of continuity? Bibliography: 2 titles.

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1. Introduction

Throughout the paper, \( \mathbb{R} \) is the set of real numbers, \( \mathbb{R}_+ \) is the set of nonnegative real numbers, and \( \mathbb{N} \) is the set of integers. For \( A \subset \mathbb{R} \) we write \( A^2 = A \times A \).

Let \( E \subset \mathbb{R}^2 \), \( 1 \leq p < \infty \). Then \( L_p(E) \) denotes the set of measurable functions \( f : E \to \mathbb{R} \) such that
\[
\|f\|_{p, E} = \left( \int_E |f|^p \right)^{1/p} < \infty.
\]
We denote by \( C(E) \) the space of uniformly continuous and bounded functions \( f : E \to \mathbb{R} \) equipped with the norm
\[
\|f\|_{\infty, E} = \sup_{E} |f|.
\]
We set \( L_\infty(E) = C(E) \). Let \( A \subset \mathbb{R} \). Then \( K(A) \) is the set of measurable functions \( \varphi : A \to \mathbb{R}_+ \) such that
\[
\int_A \varphi = 1.
\]
For \( \varphi \in K(A) \) and \( r \in \mathbb{N} \) we set
\[
\alpha_r(\varphi)_F = \left| \int_F t^r \varphi(t)dt \right|, \quad \beta_r(\varphi)_F = \int_F |t|^r \varphi(t)dt,
\]
where \( A \) and \( F \) are intervals in \( \mathbb{R} \) and \( F \subset A \).

Suppose that \( r \in \mathbb{N} \), \( u, v, x, y \in \mathbb{R} \), and \( f : \mathbb{R}^2 \to \mathbb{R} \). Then
\[
\Delta^r_{u, v}(f; x, y) = \sum_{i,j=1}^{r} (-1)^{i+j} C_i^r C_j^r f(x + iu, y + jv) - f(x, y), \quad (1.1)
\]
\[
\delta^{2r}_{u, v}(f; x, y) = \sum_{i,j=0}^{2r} (-1)^{i+j} C_i^{2r} C_j^{2r} f(x - (r-i)u, y - (r-j)v) - (C_{2r})^2 f(x, y). \quad (1.2)
\]

Remark 1. If \( f : \mathbb{R}_+^2 \to \mathbb{R} \) and \( u, v \in \mathbb{R}_+ \), then the finite difference (1.1) is also defined for such functions \( f \).

Suppose that \( 1 \leq p \leq \infty \), \( E \subset \{ \mathbb{R}_+^2, \mathbb{R}^2 \} \), \( f \in L_p(E) \), \( r \in \mathbb{N} \), and \( h_1, h_2 \geq 0 \). Then
\[
\omega_r(f; h_1, h_2)_{p, E} := \sup_{0 \leq u \leq h_1, \ 0 \leq v \leq h_2} \| \Delta^r_{u, v}(f) \|_{p, E}. \quad (1.3)
\]
In the case \( E = \mathbb{R}^2 \), we set
\[
\Omega_{2r}(f; h_1, h_2)_{p, E} := \sup_{0 \leq u \leq h_1, \ 0 \leq v \leq h_2} \| \delta^{2r}_{u, v}(f) \|_{p, E}. \quad (1.4)
\]
Note that (1.3) and (1.4) are called the moduli of continuity of \( f \) of the corresponding order.

Suppose that \( \varphi_n \in K(E_1) \), \( \psi_m \in K(E_2) \), and \( f \in L_p(\mathbb{R}^2) \). We set
\[
V^2_{n, m}(f; x, y) = \int_{E_1} \int_{E_2} \delta^2_{u, v}(f; x, y) \varphi_n(u) \psi_m(v) du dv.
\]
In this paper, we establish results similar to the following assertion.