NONMAXIMAL DECIDABLE STRUCTURES

A. Bes* and P. Cégielski†

UDC 510.665

Given any infinite structure \( \mathfrak{M} \) with a decidable first-order theory, we give a sufficient condition in terms of the Gaifman graph of \( \mathfrak{M} \) that ensures that \( \mathfrak{M} \) can be expanded with some nondefinable predicate in such a way that the first-order theory of the expansion is still decidable. Bibliography: 10 titles.

1. Introduction

Elgot and Rabin ask in [3] whether there exist maximal decidable structures, i.e., structures \( \mathfrak{M} \) with a decidable elementary theory and such that the elementary theory of any expansion of \( \mathfrak{M} \) by a nondefinable predicate is undecidable.

Soprunov proved in [10] (using a forcing argument) that every structure in which a regular ordering is interpretable is not maximal. A partial ordering \((B, <)\) is said to be regular if for every \( a \in B \) there exist distinct elements \( b_1, b_2 \in B \) such that \( b_1 < a, b_2 < a \), and no element \( c \in B \) satisfies both \( c < b_1 \) and \( c < b_2 \). As a corollary, he also proved that there is no maximal decidable structure if we replace “elementary theory” by “weak monadic second-order theory.”

In [1], we considered a weakening of the Elgot–Rabin question, namely, the question of whether all structures \( \mathfrak{M} \) whose first-order theory is decidable can be expanded by some constant in such a way that the resulting structure still has a decidable theory. We answered this question negatively by proving that there exists a structure \( \mathfrak{M} \) whose monadic second-order theory is decidable and such that any expansion of \( \mathfrak{M} \) by a constant has an undecidable elementary theory.

In this paper, we address the initial Elgot–Rabin question, and provide a criterion for nonmaximality. More precisely, given any structure \( \mathfrak{M} \) with a decidable first-order theory, we give in Sec. 3 a sufficient condition in terms of the Gaifman graph of \( \mathfrak{M} \) that ensures that \( \mathfrak{M} \) can be expanded with some nondefinable predicate in such a way that the first-order theory of the expansion is still decidable. The condition is the following: for every natural number \( r \) and every finite set \( X \) of elements of the base set \( |\mathfrak{M}| \) of \( \mathfrak{M} \), there exists an element \( x \in |\mathfrak{M}| \) such that the Gaifman distance between \( x \) and every element of \( X \) is greater than \( r \). This condition holds, e.g., for the structure \( (\mathbb{N}, S) \), where \( S \) denotes the graph of the successor function, and, more generally, for any labelled infinite graph with finite degree whose elementary theory is decidable, i.e., any structure \( \mathfrak{M} = (V, E, P_1, \ldots, P_n) \) where \( V \) is infinite, \( E \) is a binary relation of finite degree, the \( P_i \)'s are unary relations, and the elementary theory of \( \mathfrak{M} \) is decidable. Unlike Soprunov’s condition, our condition expresses some limitation on the expressive power of the structure \( \mathfrak{M} \).

In Sec. 2, we recall some important definitions and results. Section 3 deals with the main theorem. We conclude the paper with related questions.

2. Preliminaries

In the sequel, we consider first-order logic with equality. We deal only with relational structures. Given a language \( \mathcal{L} \) and an \( \mathcal{L} \)-structure \( \mathfrak{M} \), we denote by \( |\mathfrak{M}| \) the base set of \( \mathfrak{M} \). For every symbol \( R \in \mathcal{L} \), we denote by \( R^{\mathfrak{M}} \) the interpretation of \( R \) in \( \mathfrak{M} \). As usual, we will often confuse symbols and their interpretation. We denote by \( FO(\mathfrak{M}) \) the first-order (complete) theory of \( \mathfrak{M} \), i.e., the set of first-order \( \mathcal{L} \)-sentences \( \varphi \) such that \( \mathfrak{M} \models \varphi \).

We say that an \( n \)-ary relation \( R \) over \( |\mathfrak{M}| \) is elementary definable (in short: definable) in \( \mathfrak{M} \) if there exists an \( \mathcal{L} \)-formula \( \varphi \) with \( n \) free variables such that \( R = \{ (a_1, \ldots, a_n) : \mathfrak{M} \models \varphi(a_1, \ldots, a_n) \} \).

We denote by \( qr(F) \) the quantifier rank of a formula \( F \), defined inductively as follows: \( qr(F) = 0 \) if \( F \) is atomic, \( qr(¬F) = qr(F), qr(F \land G) = \max(qr(F), qr(G)) \) for \( \alpha \in \{ \land, \lor, \to \} \), and \( qr(∃x F) = qr(∀x F) = qr(F) + 1 \). We define \( FO_n(\mathfrak{M}) \) as the set of \( \mathcal{L} \)-sentences \( F \) such that \( qr(F) \leq n \) and \( \mathfrak{M} \models F \).

We say that the elementary diagram of a structure \( \mathfrak{M} \) is computable if there exists an injective map \( f : |\mathfrak{M}| \to \mathbb{N} \) such that \( \text{range} f \), as well the relations \( \{(f(a_1), \ldots, f(a_n)) : a_1, \ldots, a_n \in |\mathfrak{M}| \text{ and } \mathfrak{M} \models R(a_1, \ldots, a_n)\} \) for every relation \( R \) of \( \mathcal{L} \), are recursive (see, e.g., [9]).

*LAICL, Université Paris-Est, France, e-mail: bes@univ-paris12.fr, cegielski@univ-paris12.fr.

†These results, and the Elgot–Rabin question itself, were brought to our attention by Semenov’s paper [8].
Let us recall useful definitions and results related to the Gaifman graph of a structure [4] (see also [6]). Let \( \mathcal{L} \) be a relational language, and let \( \mathcal{M} \) be an \( \mathcal{L} \)-structure. The *Gaifman graph of \( \mathcal{M} \)*, which we denote by \( G(\mathcal{M}) \), is the undirected graph with vertex set \( [\mathcal{M}] \) such that for all \( x, y \in [\mathcal{M}] \) there is an edge between \( x \) and \( y \) if and only if \( x = y \) or there exist some \( n \)-ary relational symbol \( R \in \mathcal{L} \) and some \( n \)-tuple \( \ell \) of elements of \( [\mathcal{M}] \) that contains both \( x \) and \( y \) and satisfies \( \ell \in R^\mathcal{M} \).

The distance \( d(x, y) \) between two elements \( x, y \in [\mathcal{M}] \) is defined as the usual distance in the sense of the graph \( G(\mathcal{M}) \). We denote by \( B_r(x) \) the \( r \)-ball with center \( x \), i.e., the set of elements \( y \) of \( [\mathcal{M}] \) such that \( d(x, y) \leq r \). It should be noted that for every fixed \( r \) the binary relation “\( y \in B_r(x) \)” is definable in \( \mathcal{M} \). For every \( X \subseteq [\mathcal{M}] \) we define \( B_r(X) \) as \( B_r(X) = \bigcup_{x \in X} B_r(x) \).

An \( r \)-local formula \( \varphi(x_1, \ldots, x_n) \) is a formula whose quantifiers are all relativized to \( B_r(\{x_1, \ldots, x_n\}) \). We will use the notation \( \varphi^{(r)} \) to indicate that \( \varphi \) is \( r \)-local.

Let us now state Gaifman’s theorem about local formulas.

**Theorem 1** ([4]). Let \( \bar{x} = (x_1, \ldots, x_n) \), and let \( \varphi(\bar{x}) \) be an \( \mathcal{L} \)-formula. From \( \varphi \) one can effectively compute a formula that is equivalent to \( \varphi \) and is a boolean combination of formulas of the form

- \( \psi^{(r)}(\bar{x}) \),
- \( \exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i \leq s} \alpha^{(r)}(x_i) \land \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \right) \),

where \( s \leq q_r(\varphi) + n \) and \( r \leq 7^k \).

Moreover, if \( \varphi \) is a sentence, then only sentences of the second kind occur in the resulting formula.

### 3. A sufficient condition for nonmaximality

The aim of this section is to prove the following theorem.

**Theorem 2.** Let \( \mathcal{L} \) be a finite relational language, and let \( \mathcal{M} \) be an infinite countable \( \mathcal{L} \)-structure that satisfies the following conditions:

1. \( \text{FO}(\mathcal{M}) \) is decidable;
2. every element of \( [\mathcal{M}] \) is definable in \( \mathcal{M} \);
3. for every finite set \( X \subseteq [\mathcal{M}] \) and every \( r \in \mathbb{N} \), there exists \( a \in [\mathcal{M}] \) such that \( d(a, X) > r \).

Then there exists a unary predicate symbol \( R \notin \mathcal{L} \) and a \( (\mathcal{L} \cup \{R\}) \)-expansion \( \mathcal{M}' \) of \( \mathcal{M} \) such that

- \( \text{FO}(\mathcal{M}') \) is decidable;
- the set \( R^{\mathcal{M}'} \) is not definable in \( \mathcal{M} \);
- the elementary diagram of \( \mathcal{M}' \) is computable.

Note that in the above theorem, the construction of \( \mathcal{M}' \) from \( \mathcal{M} \) can be repeated starting from \( \mathcal{M}' \). Indeed, \( \mathcal{M}' \) clearly satisfies Conditions (1) and (2). Moreover, expanding a structure by unary predicates does not modify its Gaifman graph, therefore we have \( G(\mathcal{M}') = G(\mathcal{M}) \), which implies that Condition (3) also holds for \( \mathcal{M}' \).

Let us illustrate Theorem 2 with a few examples.

- The structure \( \mathcal{M} = (\mathbb{N}; S) \), where \( S \) denotes the graph of the function \( x \mapsto x + 1 \), satisfies all conditions of Theorem 2. Indeed, Langford [5] proved that \( \text{FO}(\mathcal{M}) \) is decidable. Moreover, Condition (2) is easy to prove, and Condition (3) is a straightforward consequence of the fact that \( d(x, y) = |x - y| \) for all natural numbers \( x, y \).
- The same holds for any structure of the form \( \mathcal{M} = (\mathbb{N}; S, P_1, \ldots, P_n) \) where the \( P_i \)‘s denote unary predicates and \( \text{FO}(\mathcal{M}) \) is decidable (the Gaifman graph of any such structure is equal to that of \( (\mathbb{N}; S) \), see the remark above).
- More generally, Theorem 2 applies to any infinite labelled graph with finite degree, more precisely, to any structure of the form \( \mathcal{M} = (V; E, P_1, \ldots, P_n) \) where \( V \) is infinite, \( E \) is a binary relation with finite degree, the \( P_i \)‘s denote unary predicates, \( \text{FO}(\mathcal{M}) \) is decidable, and every element of \( V \) is definable in \( \mathcal{M} \). In this case, the Gaifman graph of \( \mathcal{M} \) has finite degree, which implies Condition (3). Note that Theorem 2 also applies to some structures for which the degree of the Gaifman graph is infinite – see the last example.
- The structure \( \mathcal{M} = (\mathbb{N}; <) \) does not satisfy Condition (3) of Theorem 2, since \( d(x, y) \leq 1 \) for all \( x, y \in \mathbb{N} \).
- Observe that \( \text{FO}(\mathcal{M}) \) is decidable [5], and, moreover, \( \mathcal{M} \) is not maximal: consider, e.g., the structure \( \mathcal{M}' = (\mathbb{N}; <, +) \) where + denotes the graph of addition; \( \text{FO}(\mathcal{M}') \) is decidable [7], and + is not definable in \( \mathcal{M} \), since in \( \mathcal{M} \) one can only define finite or co-finite subsets of \( \mathbb{N} \).