FUZZY CONSTRUCTIVE LOGIC

I. D. Zaslavsky*  

We introduce a logical system in which the principles of fuzzy logic are interpreted from the point of view of the constructive approach. The language of predicate formulas without functional symbols and symbols of constants is considered. The notion of identically true predicate formula in the framework of the introduced logic is defined; two variants of this definition are given. Theorems concerning identically true predicate formulas are proved. Some connections between the introduced logic and the constructive (intuitionistic) predicate calculus are established. Bibliography: 40 titles.

1. Introduction

The research reflected in this paper was conducted with two main purposes. The first of them is to develop a constructive version of fuzzy logic; the second one is to establish new relations between fuzzy logic and the traditional machinery of the predicate calculus. As is well known, fuzzy logic, created by L. Zadeh [39], has recently gained wide popularity and presently has important practical applications (see [32, 38]). Taking into account the universal character of the constructive (see [9, 11, 12, 16, 17]) and intuitionistic (see [19, 22, 25]) approaches, we may hope that the development of a constructive version of fuzzy logic will provide a clearer understanding of the algorithmic aspects of the notions of fuzzy logic and their algorithmic essence. Establishing relations between fuzzy logic and the traditional machinery of the predicate calculus may become the starting point for constructing such calculi as “fuzzy constructive propositional calculus” and “fuzzy constructive predicate calculus,” which would allow one to formalize the main reasoning techniques applied in fuzzy logic (note that a number of important results in this direction were obtained in [21]). We may hope that in this way it will be possible to construct formal axiomatic theories based on fuzzy logic, for example, “fuzzy formal arithmetic.” This would allow, in particular, to solve the problem posed by P. K. Rashevskii [14], namely, the problem of constructing a theory of “fuzzy positive integers”: such a theory, in P. K. Rashevskii’s opinion, would be related to some open problems of experimental sciences (e.g., physics).

In this paper, we introduce a logical system, fuzzy constructive logic (in what follows, abbreviated to FCL), in which the basic principles of fuzzy logic are combined with the constructive approach. This system has a number of common features with “continuous constructive logic” described in [1]; however, it is developed by other methods. In particular, the semantics of FCL is based on a certain deductive metasystem. In the existing definitions of semantics of logical systems, one usually does not describe the metasystem underlying the corresponding definition; but this metasystem often affects the properties of the semantics in some or other way. For example, in S. C. Kleene’s semantics for intuitionistic logic (see [23, 24]), the so-called “Gene Rose’s effect” holds: there exist formulas that are unprovable in the intuitionistic propositional calculus, but identically true according to S. C. Kleene’s definitions [34]. However, the existence of such an effect substantially depends on the metasystem on which the semantics is based: if we take as a metasystem the constructive logical-mathematical system without A. A. Markov’s principle (see [3, 10]), then Gene Rose’s theorem [34] does not work, and the very existence of “Gene Rose’s effect” is open to question. (This fact was brought to the author’s attention by A. G. Dragalin.) In view of these observations, the author thinks that explicitly introducing the metasystem into semantic considerations is in no way unnatural.

In Sec. 2 of the paper, we define some notions related to fuzzy recursively enumerable sets; these sets play the role of an algorithm-theoretic base for further constructions. In Sec. 3, we introduce the language of predicate formulas in the framework of which we define the semantics of fuzzy constructive logic. Section 4 contains definitions of the metasystems serving as a base for semantic considerations. In Sec. 5, we define the FCL-validity and FCL-quasivalidity of predicate formulas; these notions are the essence of the semantics of fuzzy constructive logic we introduce (it will sometimes be called the “FCL-semantics”). In Sec. 6, we establish the FCL-validity and FCL-quasivalidity of some predicate formulas. We also show that some predicate formulas are not FCL-valid and FCL-quasivalid; these assertions hold for all metasystems that we consider as a base for the semantics of predicate formulas. It turns out that all formulas whose FCL-validity or FCL-quasivalidity is established below are provable in the constructive (intuitionistic) predicate calculus; on the contrary, the formulas that turn out

*Institute of Informatics and Automation Problems, Armenian Academy of Sciences, Yerevan, Armenia, e-mail: zaslavn@mail.ru.

to be non-FCL-quasivalid are not provable in this calculus. Thus the semantics of predicate formulas under consideration is to a certain extent related to provability in the constructive (intuitionistic) predicate calculus. At the end of the paper, we outline some perspectives for further research.

We assume that the reader is familiar with the theory of recursive functions (see [4, 24, 26, 33]) and the machinery of classical and traditional constructive (intuitionistic) predicate logic (see [20, 24, 31, 36, 37]). As PRF, ParRF, TRF, RES, PRP, REP we abbreviate the terms “primitive recursive function,” “partial recursive function,” “total recursive function,” “recursively enumerable function,” “primitive recursive predicate,” “recursively enumerable predicate,” respectively. All further considerations are in the framework of the constructive approach as it is adopted in A. A. Markov’s school (see [9, 11, 12, 16, 17]). A. A. Markov’s principle (see [3, 10]) will not be applied in what follows.

2. Fuzzy recursively enumerable sets

The basic algorithmic notion of fuzzy constructive logic is the notion of a fuzzy recursively enumerable set. It was introduced by the author in [1] and studied in [5-8, 27-30]. Recall the corresponding definitions. A fuzzy recursively enumerable set (in what follows, FRES) of dimension \( n \geq 1 \) is a recursively enumerable set of systems of the form \((x_1, x_2, \ldots, x_n, \varepsilon)\), where all \( x_i \) are positive integers and \( \varepsilon \) is a binary rational of the form \( \frac{k}{2^n} \) such that \( 0 \leq \varepsilon \leq 1 \). The notion of FRES can be regarded as an algorithmic version of the notion of a fuzzy set, the basic notion of fuzzy logic (see [32, 38, 39]). The interrelations between the notion of FRES and the existing concepts of fuzzy logic are considered in [8] and [30] (see [8, pp. 86-89; 30, p. 4002]).

Let us introduce some auxiliary notions related to FRES. We assume that for each \( n \geq 1 \) there is a fixed constructive one-to-one correspondence between the set of systems \((x_1, x_2, \ldots, x_n, \varepsilon)\) of the form described above and the set of nonnegative integers \( N = \{0, 1, 2, \ldots\} \). The nonnegative integer corresponding to a system \((x_1, x_2, \ldots, x_n, \varepsilon)\) is denoted by \( \Gamma_n(x_1, x_2, \ldots, x_n, \varepsilon) \); the system corresponding to a nonnegative integer \( k \) is denoted by \( \Gamma_n^{-1}(k) \). If \( \alpha \) is a FRES of dimension \( n \), then the RE-image of \( \alpha \) is the one-dimensional RES \( \beta \) containing all nonnegative integers of the form \( \Gamma_n(x_1, x_2, \ldots, x_n, \varepsilon) \) for \((x_1, x_2, \ldots, x_n, \varepsilon) \in \alpha \) (and only them). If \( \beta \) is any one-dimensional RES and \( n \) is any positive integer, \( n \geq 1 \), then the \( n \)-dimensional FRES-preimage of \( \beta \) is the FRES \( \alpha \) of dimension \( n \) containing all systems of the form \( \Gamma_n^{-1}(k) \) for \( k \in \beta \) (and only them). Clearly, any one-dimensional RES \( \beta \) has infinitely many FRES-preimages, for \( n = 1, 2, \ldots \); a FRES \( \alpha \) is uniquely determined by its dimension \( n \) and its RE-image \( \beta \).

For every FRES \( \alpha \) of dimension \( n \), we define the Specker function \( \Psi_\alpha \) via the algorithm that, given a system \((x_1, x_2, \ldots, x_n)\) of positive integers, outputs the supremum of all \( \varepsilon \) satisfying the condition \((x_1, x_2, \ldots, x_n, \varepsilon) \in \alpha \) (and 0 if there are no such \( \varepsilon \)). It is easy to see that this supremum can always be obtained constructively as a so-called Specker number, i.e., a pseudonumber \( \mathbb{N} \) determined by a constructive nondecreasing bounded sequence of binary rationals (see [3, 35]). As is well known, not every Specker number can be given as a constructive real number (see [3, 35]); however, the Gödel numbering of Specker numbers, as well as the ordering and equality relations between them can be defined in an obvious way by analogy with constructive real numbers [3]. It is easy to see that for every FRES \( \alpha \) of dimension \( n \), one can construct the TRF that, given a system \((x_1, x_2, \ldots, x_n)\), outputs the Gödel number of the Specker number \( \Psi_\alpha(x_1, x_2, \ldots, x_n) \). Clearly, the inequalities \( 0 \leq \Psi_\alpha(x_1, x_2, \ldots, x_n) \leq 1 \) always hold. Note that Specker functions for FRES were studied in [27].

An \( n \)-dimensional FRES \( \alpha \) is called open if it satisfies the following conditions:

1. all systems of the form \((x_1, x_2, \ldots, x_n, 0)\) belong to \( \alpha \);
2. if \((x_1, x_2, \ldots, x_n, \varepsilon) \in \alpha \) and \( 0 \leq \delta < \varepsilon \), then \((x_1, x_2, \ldots, x_n, \delta) \in \alpha \);
3. for any system \((x_1, x_2, \ldots, x_n, \varepsilon) \in \alpha \) with \( \varepsilon > 0 \), there is \( \delta > \varepsilon \) such that \((x_1, x_2, \ldots, x_n, \delta) \in \alpha \).

Now let \( \Psi \) be an algorithm such that for any system \((x_1, x_2, \ldots, x_n)\) of positive integers, \( \Psi(x_1, x_2, \ldots, x_n) \) is a Specker number satisfying the condition \( 0 \leq \Psi(x_1, x_2, \ldots, x_n) \leq 1 \) (we assume that the TRF determining this algorithm, given the system \((x_1, x_2, \ldots, x_n)\), outputs the Gödel number of this Specker number). The standard realization of the algorithm \( \Psi \) is the FRES \( \beta \) of dimension \( n \) such that \((x_1, x_2, \ldots, x_n, \varepsilon) \in \beta \) if and only if \( \varepsilon = 0 \) or \( \varepsilon < \Psi(x_1, x_2, \ldots, x_n) \). It is easy to check that the FRES \( \beta \) constructed in this way is always open and satisfies the condition

\[ \Psi_\beta(x_1, x_2, \ldots, x_n) = \Psi(x_1, x_2, \ldots, x_n). \]

Obviously, there is a constructive way to pass from a FRES to the corresponding Specker function and, conversely, from a given Specker function to the corresponding FRES.

The open image \( O(\alpha) \) of a FRES \( \alpha \) is the FRES \( \beta \) that is the standard realization of the function \( \Psi_\alpha \).