PYRAMIDS IN THREE-DIMENSIONAL NORMED SPACES

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It is proved that vertex sets of Euclidean tetrahedra of special type can be isometrically embedded in an arbitrary three-dimensional normed space. In particular, this holds true for every regular triangular pyramid. Bibliography: 9 titles.

Dvoretsky’s familiar theorem [2] easily implies that the vertex set of any Euclidean simplex Δ can be isometrically embedded in any real normed space of sufficiently large dimension. (For regular simplices Δ, see [1].)

Does this dimension depend only on the dimension of Δ? It is proved in [5] that each three-dimensional normed space contains four points lying at equal distances from each other. Similarly, it is proved in [6] that any four-dimensional normed space contains five equidistant points. Nothing seems to contradict the assumption that the vertex set of any Euclidean tetrahedron can be isometrically embedded in any three-dimensional normed space. Below, we prove this assertion in certain special cases.

§1. TWO-DIMENSIONAL CASE

We will need the following lemma, which describes possibility of realizing a Euclidean triangle with sides of length a, b, and c on a two-dimensional normed plane with strictly convex unit disk.

Lemma 1. Let $E$ be a two-dimensional normed plane with strictly convex unit disk, and let $A, B \in E$ be two points with $AB = c$. Then $E$ contains exactly two points $C$ such that $AC = b$ and $BC = a$.

Proof. The existence of such points $C$ follows from simple continuity arguments. Choosing the point $C$ on the line $AB$ at a distance of $b$ from the point $A$, and rotating the segment $CA$ through the angle of $\pi$ about the point $A$, we see that $BC$ attains all values from $b - c$ to $b + c$. Thus, each half-plane of $E$ with boundary $AB$ contains the required point $C$.

In order to prove that each of the two half-planes with boundary $AB$ contains a unique point $C$ with required properties, we argue by contradiction. Assume that two points $C$ and $C_1$ on one side of $AB$ possess the required property. We plot vectors $BC$ and $BC_1$ equal to the vectors $AC$ and $AC_1$, respectively. Then $CC_1$ and $C'C_1$ are equal parallel chords of two concentric disks in $E$. They lie on one side of $AB$ and are seen from the center of the disks at two angles, none of which is contained in the other, which is impossible.

We observe that the normed plane where the unit disk is a parallelogram $P$ contains a continuum of equilateral triangles with one side coinciding with a side of $P$.

§2. TRIANGULAR PYRAMIDS IN NORMED SPACE

Theorem 1. Let $E$ be a three-dimensional normed space. Let $\Delta$ be a Euclidean tetrahedron where four edges in two pairs of skew edges have length $a > 0$, one of the remaining two edges has length $b$, where $b < a$, and the sixth edge has an arbitrary length in the interval $(0; 2a - b)$. Then the vertex set of $\Delta$ can be isometrically embedded in $E$ so that one of the faces of $\Delta$ with edges of length $a$ and $b$ lies in a prescribed plane in $E$, and the edge of length $b$ has prescribed direction.

Proof. We prove the theorem in the case where the space $E$ has strictly convex unit ball $B$. In the remaining cases, the theorem is proved by passing to the limit.

Let $AB$ be a segment of length $b$ lying in a prescribed plane $P$ in $E$ and having a prescribed direction. Since the ball $B$ is strictly convex, Lemma 1 implies that the plane $P$ contains a unique positively oriented triangle $ABC$ with $AC = BC = a$. We rotate the plane $P$ about the line $AB$ and consider the corresponding triangles $ABC$. After rotation of $P$ through 180°, the vertex $C$ of the triangle once more takes position $C_1$ in the plane $P$. Furthermore, the quadrangle $ACBC_1$ is a rhombus, where the sides have length $a$ and the diagonal $AB$ has length $b$. Consequently, the length of the diagonal $CC_1$ is at least $2a - b$ by the triangle inequality. Now the existence of the required isometric embedding follows from simple continuity arguments.

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**Theorem 2.** Each three-dimensional normed space \( E \) contains an isometric copy of the vertex set of any Euclidean tetrahedron where four edges in two pairs of skew edges have length \( a > 0 \), one of the remaining edges has length \( \sqrt{3}a \), the sixth edge attains any value in \( (0, \sqrt{3}a) \), and, finally, one of the faces with edges of length \( a \) and \( \sqrt{3}a \) lies in a prescribed plane in \( E \).

**Proof.** Let \( P \) be the prescribed plane. The proof of Theorem 1 shows that it suffices to find a rhombus in \( P \) with sides of length \( a \) and equal diagonals of length \( \sqrt{3}a \). In order to prove that there exists such a rhombus, we use the fact that there exists an affine-regular octagon \( A_1, A_2, \ldots, A_8 \) inscribed in the unit disk on the plane \( P \) (see [3]). Then \( A_1, A_2, \ldots, A_7 \) is the required rhombus. \( \square \)

**Theorem 3.** The vertex set of any regular Euclidean triangular pyramid can be isometrically embedded in an arbitrary three-dimensional normed space

**Proof.** As before, we assume that the ball \( B \subset E \) is strictly convex.

Our proof involves the following result.

**Lemma 2.** Any three-dimensional normed space contains equilateral triangles with medians of equal length.

**Proof.** We denote by \( M \) the space of positively oriented equilateral triangles with sides of unit length and barycenter at a prescribed point of the normed space. Lemma 1 implies that \( M \) is a manifold homeomorphic to the group \( SO(3) \) of rotations. The cyclic group \( \mathbb{Z}_a \) acts on \( M \) and \( \mathbb{R}^3 \) by cyclic permutations of vertices of the triangles and coordinates of points, respectively.

Consider the continuous mapping

\[
F: M \to \mathbb{R}^3: \triangle A_1 A_2 A_3 \mapsto (AM_1, AM_2, AM_3),
\]

where \( AM_1, AM_2, \) and \( AM_3 \) are medians of the triangle \( A_1 A_2 A_3 \). By construction, the mapping \( F \) is \( \mathbb{Z}_a \)-equivariant. Results of [4] and [7] imply that in this case the image of \( F \) contains a point with equal coordinates, which fact proves Lemma 2. \( \square \)

We observe that, generally speaking, a two-dimensional plane contains no equilateral triangles with medians of equal length.

Lemma 2 easily implies the following result.

**Corollary.** Any three-dimensional normed space contains equilateral triangles with sides of prescribed length \( a > 0 \) and equal medians of length at least (or, optionally, at most) \( \sqrt{3}a/2 \).

**Proof.** Indeed, it is well known that the median vectors of any triangle \( T \) constitute a triangle \( T' \), while the median vectors of \( T' \) are proportional to the side vectors of \( T \) with factor \( 3/4 \). If the initial triangle \( T \) is equilateral and has medians of equal length, then the same holds true for the triangle \( T' \). Furthermore, for one of the triangles the ratio of the length of the medians to the length of the sides is at most \( \sqrt{3}/2 \), while for the other one the ratio is at least \( \sqrt{3}/2 \). \( \square \)

**End of the proof of Theorem 3.** Let \( T \) be an equilateral triangle lying in a plane \( P \subset E \). We assume that the medians of \( T \) have equal length, which is at least \( \sqrt{3}/2 \) times the length of the side of \( T \).

We consider the central section of the ball \( B \subset E \) by a plane \( P \parallel P \), and inscribe in \( P \cap B \) a triangle \( T' \) positively homothetic to \( T \). Since the section is strictly convex, such a triangle \( T' \) is unique, and the barycenter of \( T' \) is the center \( O \) of \( B \).

Let \( P_2 \) be the three-plane of support of \( B \parallel P \).

Consider all possible positive homothets of the triangle \( T \) that are inscribed in sections of the form \( B \cap P_3 \), where \( P_3 \) is a plane parallel to \( P_1 \) and \( P_2 \) and lying between them. We observe that since the sections are strictly convex, the considered homothets of \( T \) are uniquely determined by and continuously depend on the plane \( P_3 \). For every such a homothet of \( T \), we consider the ratio \( \|O A_1\|/\|A_1 A_2\| \), where \( O \) is the center of the ball \( B \). If \( T \) is inscribed in the central section \( B \cap P_1 \), then we have

\[
\frac{\|O A_1\|}{\|A_1 A_2\|} \leq \frac{1}{\sqrt{3}}
\]

If the secant plane \( P_3 \) is sufficiently close to \( P_2 \), then for homothets of \( T \) inscribed in \( P_3 \cap B \) we have \( \|O A_1\|/\|A_1 A_2\| > x \) for a prescribed \( x > 0 \). Therefore, continuity arguments imply that the ratio attains...