HOCHSCHILD COHOMOLOGY FOR SELF-INJECTIVE ALGEBRAS OF TREE CLASS $D_n$. II

Yu. V. Volkov

UDC 512.5

The minimal projective bimodule resolution for a certain family of representation-finite self-injective algebras of tree class $D_n$ is constructed. The dimensions of the groups of Hochschild cohomology for the algebras under consideration are calculated by the instrumentality of this resolution. The resolution constructed is periodic, and accordingly the Hochschild cohomology for these algebras is periodic as well. Bibliography: 12 titles.

1. INTRODUCTION

Let $R$ be a representation-finite self-injective basic algebra over an algebraically closed field $k$. The stable $AR$-quiver of such an algebra can be described by an associate tree and this tree is one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$ (see [1]). If for the algebra $R$ this tree is $A_n$, then by [2] the algebra $R$ is stably equivalent either to a serial self-injective algebra or to the so-called “Möbius algebra.” In [3], the algebra of Hochschild cohomology $HH^*(R)$ for serial self-injective algebras was calculated and for the Möbius algebra in [4] the subalgebra $HH^*(R)$ of the algebra $HH^*(R)$ that is generated by homogeneous elements of degree divisible by $r$, where $r$ is a parameter associated with the defining relations for the algebra $R$, was derived. In these two papers, the fact that a syzygy of an appropriate order for the $R$-bimodule $R$ can be described as a twisted bimodule, was essentially used. A more direct approach to the calculation of the Hochschild cohomology for the Möbius algebra $R$ was proposed in [5], where the minimal projective resolution of the algebra $R$ as a $\Lambda$-module, where $\Lambda$ is the enveloping algebra for $R$, was constructed. Further in [6] this resolution was used for calculation of the additive structure for the algebra $HH^*(R)$, i.e., for the Möbius algebra $R$, the dimensions of the groups $HH^*(R)$ were calculated.

If for the algebra $R$ this associate tree is $D_n$, then, by [7], the algebra $R$ is stably equivalent to an algebra of one of the five types. Their quivers with relations were represented in the same article. In [8], using the approach of [5], the periodic minimal projective resolution for algebras of one of these types was constructed. Further it was used for calculation of the additive structure for the algebra $HH^*(R)$. In the present paper, the periodic bimodule resolution for another type of algebras of tree class $D_n$ is constructed (see Theorem 1). Then we give a description of the additive structure for the algebra $HH^*(R)$, which was obtained with the help this resolution (see Theorems 2–4).

Note that the periodicity of the minimal bimodule resolution for standard representation-finite self-injective algebras was proved in [11] for all cases, except for a series of algebras which are dealt with in the present paper. Finally, recently the periodicity of the minimal bimodule resolution for all representation-finite self-injective algebras was proved (see [12]).

2. BIMODULE RESOLUTION

Assume that $k$ is an algebraically closed field. Let $r$, $n \in \mathbb{N}$, $3 \nmid r$, $n \geq 2$ be fixed. Consider the following quiver with relations $(Q, I)$. The set of vertices is $Q_0 = \{i \in \mathbb{N} \mid 1 \leq i \leq rn\}$; further the elements of $Q_0$ are specified modulo $rn$. The set of arrows $Q_1$ for the quiver $Q$ consists of the following elements:

$$\alpha_{s,i} : (s-1)n + i \to (s-1)n + i + 1,$$

*St.Petersburg State University, St.Petersburg, Russia, e-mail: wolf86@list.ru.


where \( s \in \{1, \ldots, r \} \), \( i \in \{1, \ldots, n-1 \} \),

\[
\gamma_s : sn \to (s+1)n+1,
\]

\[
\beta_s : sn \to (s+1)n,
\]

where \( s \in \{1, \ldots, r \} \). Henceforth the index \( s \) for \( \alpha_{s,i}, \beta_s \) and \( \gamma_s \) is specified modulo \( r \).

The ideal \( I \) is generated by the following elements of the path algebra \( kQ \) of the quiver \( Q \):

\[
\gamma_s \alpha_{s,n-1},
\]

\[
\beta_{s+1} \beta_s - \alpha_{s+2,n-1} \ldots \alpha_{s+2,1} \gamma_s,
\]

\[
\alpha_{s+3,t} \ldots \alpha_{s+3,1} \gamma_{s+1} \beta_s \alpha_{s,n-1} \ldots \alpha_{s,t},
\]

where \( s = 1, \ldots, r \) and \( t = 1, \ldots, n-2 \). It is easy to check that the last element for \( t = n-1 \), \( s = 1, \ldots, r \) lies in the ideal \( I \) as well. Consider the \( k \)-algebra \( R = kQ/I \). It follows from \([7]\) that the tree class of \( R \) is \( D_{3n} \). Denote by \( \Lambda = R \otimes R^\oplus \) the enveloping algebra of the algebra \( R \). Denote by \( e_{i,j} \) \((1 \leq i \leq r \) and \( 1 \leq j \leq n \)) the idempotent of the algebra \( R \) corresponding to the vertex \((i-1)n+j\) of the quiver \( Q \). In the sequel, we take the first index in \( e_{i,j} \) modulo \( r \). Then \( \{e_{i,j} \otimes e_{i',j'} \} \) \( i, j, i', j' \) is a full set of orthogonal primitive idempotents for the algebra \( \Lambda \). Denote by \( P_{i,j} = Re_{i,j} \) the projective \( R \)-module corresponding to the vertex \((i-1)n+j\) of the quiver \( Q \) and denote by \( S_{i,j} \) the simple \( R \)-module corresponding to it. Denote by \( P_{i-1,j} \otimes e_{i,j} \otimes e_{i',j} \) the projective \( R \)-module corresponding to the idempotent \( e_{i,j} \otimes e_{i',j} \). If \( w \) is a path in the quiver \( Q \) starting at the vertex \((i_1-1)n+j_1 \) and ending at the vertex \((i_2-1)n+j_2 \), then the right multiplication by \( w \) induces a homomorphism \( u^* : P_{i_2,j_2} \to P_{i_1,j_1} \). We denote it by \( w \). If \( w_1 \) is a path starting at the vertex \((i_1-1)n+j_1 \) and ending at the vertex \((i_2-1)n+j_2 \) and \( w_2 \) is a path starting at the vertex \((i_1-1)n+j_1 \) and ending at the vertex \((i_2-1)n+j_2 \) then \( w_1 \otimes w_2 \in \text{Hom}_A(P_{i_2,j_2} \otimes e_{i,j}, P_{i_1,j_1} \otimes e_{i,j}) \).

We introduce also the auxiliary notation

\[
\tau_i = \gamma_{i+1} \beta_i, \quad \mu_{i,j} = \alpha_{i,j} \ldots \alpha_{i,1}, \quad \nu_{i,j} = \alpha_{i,n-1} \ldots \alpha_{i,j}.
\]

**Remark 1.** Henceforth for the uniformity of notation we suppose additionally that the empty product of the quiver’s arrows is identified with an appropriate idempotent of the algebra \( R \); for example, \( \mu_{i,0} = e_{i,1}, \alpha_{i,j-1} \ldots \alpha_{i,j} = e_{i,j}, \nu_{i,n-2} = e_{i,n-2} \).

It is clear that the set

\[
B_R = \{ \mu_{i+3,t-1} \tau_i \nu_{i,j} \mid 1 \leq i \leq r, 1 \leq t \leq n \} \cup \{ \alpha_{i,t-1} \ldots \alpha_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq t \leq n \}
\]

\[
\cup \{ \beta_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq n \} \cup \{ \mu_{i+2,j-1} \gamma_i \mid 1 \leq i \leq r, 1 \leq j \leq n \}
\]

constitutes a \( k \)-basis for the algebra \( R \). We say that \( B_R \) is the standard basis for the algebra \( R \).

It is clear that \( B_\Lambda = \{ u \otimes v \mid u, v \in B_R \} \) constitutes a \( k \)-basis for the algebra \( \Lambda \). We say that \( B_\Lambda \) is the standard basis for the algebra \( \Lambda \). It is clear that \( B_{[i_1,j_1][i_2,j_2]} = B_\Lambda \cap P_{[i_1,j_1][i_2,j_2]} \) constitutes a \( k \)-basis