Let $\Phi$ be a real valued function of one real variable, let $L$ denote an elliptic second order formally self-adjoint differential operator with bounded measurable coefficients, and let $P$ stand for the Poisson operator for $L$. A necessary and sufficient condition on $\Phi$ ensuring the equivalence of the Dirichlet integrals of $\Phi \circ Ph$ and $P(\Phi \circ h)$ is obtained. We illustrate this result by some sharp inequalities for harmonic functions. Bibliography:

1. Introduction

In the present article, we consider an elliptic second order formally self-adjoint differential operator $L$ in a bounded domain $\Omega$. We denote by $Ph$ the $L$-harmonic function with the Dirichlet data $h$ on $\partial \Omega$. The Dirichlet integral corresponding to the operator $L$ will be denoted by $D[u]$. We also introduce a real-valued function $\Phi$ on the line $\mathbb{R}$ and denote by $\Phi \circ u$ the composition of $\Phi$ and $u$.

We want to show that the Dirichlet integrals of the functions $\Phi \circ Ph$ and $P(\Phi \circ h)$ are comparable. First of all, clearly, the inequality

$$D[P(\Phi \circ h)] \leq D[\Phi \circ Ph]$$

is valid. Hence we only need to check the opposite estimate

$$D[\Phi \circ Ph] \leq C D[P(\Phi \circ h)]. \quad (1.1)$$

We find a condition on $\Phi$ which is both necessary and sufficient for (1.1).

* To whom the correspondence should be addressed.

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Moreover, we prove that the two Dirichlet integrals are comparable if and only if the derivative \( \Psi = \Phi' \) satisfies the reverse Cauchy inequality

\[
\frac{1}{b-a} \int_a^b \Psi^2(t) \, dt \leq C \left( \frac{1}{b-a} \int_a^b \Psi(t) \, dt \right)^2
\]

for any interval \((a, b) \subset \mathbb{R}\).

We add that the constants \( C \) appearing in (1.1) and (1.2) are the same.

At the end of the paper, this result is illustrated for harmonic functions and for \( \Psi(t) = |t|^\alpha \) with \( \alpha > -1/2 \). In particular, we obtain the sharp inequalities

\[
\int_\Omega |\nabla (|Ph| Ph)|^2 \, dx \leq \frac{3}{2} \int_\Omega |\nabla P(|h| h)|^2 \, dx ,
\]

(1.3)

\[
\int_\Omega |\nabla (Ph)^3|^2 \, dx \leq \frac{9}{4} \int_\Omega |\nabla P(h^3)|^2 \, dx
\]

(1.4)

for any \( h \in W^{1/2,2}(\partial \Omega) \) and

\[
\int_\Omega |\nabla (Ph)^2|^2 \, dx \leq \frac{4}{3} \int_\Omega |\nabla P(h^2)|^2 \, dx ,
\]

(1.5)

\[
\int_\Omega |\nabla (Ph)^3|^2 \, dx \leq \frac{9}{5} \int_\Omega |\nabla P(h^3)|^2 \, dx
\]

(1.6)

for any nonnegative \( h \in W^{1/2,2}(\partial \Omega) \). Here, \( P \) is the harmonic Poisson operator.

To avoid technical complications connected with non-smoothness of the boundary, we only deal with domains bounded by surfaces of class \( C^\infty \), although, in principle, this restriction can be significantly weakened.

2. Preliminaries

All functions in this article are assumed to take real values and the notation \( \partial_i \) stands for \( \partial/\partial x_i \).

Let \( L \) be the second order differential operator

\[
Lu = -\partial_i (a_{ij}(x) \partial_j u)
\]

defined in a bounded domain \( \Omega \subset \mathbb{R}^n \).

The coefficients \( a_{ij} \) are measurable and bounded. The operator \( L \) is uniformly elliptic, i.e., there exists \( \lambda > 0 \) such that

\[
a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2
\]

(2.1)

for all \( \xi \in \mathbb{R}^n \) and for almost every \( x \in \Omega \).