ADIABATIC ALMOST-PERIODIC SCHRÖDINGER OPERATORS

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In this review, spectral results are described for a one-dimensional almost-periodic Schrödinger operator with two frequencies, one of them being much greater than the other. Bibliography: 36 titles.

1. INTRODUCTION

We discuss the ergodic operator family

\[ H = -\frac{d^2}{dx^2} + V(x-z) + W(\varepsilon x), \quad x \in \mathbb{R}, \quad (1.1) \]

where \( V : \mathbb{R} \to \mathbb{R} \) is a nonconstant, locally square integrable, 1-periodic function,

\[ V(x+1) = V(x), \quad x \in \mathbb{R}; \quad (1.2) \]

0 < \varepsilon < 2\pi is the \textit{frequency}, a constant such that the ratio 2\pi/\varepsilon is irrational; 0 < z < 2\pi is the \textit{ergodic} parameter; \( W \) is a nonconstant \( 2\pi \)-periodic function analytic in a neighborhood of the real line and taking real values along the real line.

The choice of the periods of \( V \) and \( W \) is convenient for the sequel.

Since the ratio 2\pi/\varepsilon is irrational, it follows that the function \( x \to V(x-z) + W(\varepsilon x) \), the potential of the Schrödinger operator, is almost periodic, and the operator is called \textit{almost periodic}. The main spectral properties of the operators of family (1.1) are the same for almost all values of the ergodic parameter; the reader will find a short review in Sec. 2.

In this paper, we discuss the \textit{adiabatic} case where the number \( \varepsilon \) is small and one of the frequencies is much greater than the other.

The spectrum of almost-periodic operators is one of the most timely fields of study in modern mathematical physics. It is motivated by the quantum physics of solid state. The reader can find reviews and lists of references in [34, 5, 27, 30, 35].

Note that more or less complete analysis was carried out only for a few one-dimensional difference Schrödinger operators with “simple” potentials (almost Mathieu operator, Maryland model, operators with potentials taking a finite number of values). For differential operators, most of the results were obtained in the case of analytic potentials of the form \( \lambda p(\omega_1 x, \omega_2 x, \ldots, \omega_n x) \), where \( p \) is a periodic (e.g., 1-periodic) function of the variables \( \omega_1 x, \omega_2 x, \ldots, \omega_n x \), the numbers \( \omega_1, \omega_2, \ldots, \omega_n \) are “strongly” incommensurate frequencies (in (1.1), there are two frequencies), and \( \lambda > 0 \) is a large or a small coupling constant. In the case of large coupling, for almost all values of the ergodic parameter, a singular spectrum was found (see, for example, [36]). In the case of small coupling, all the spectrum proves to be absolutely continuous and the generalized eigenfunctions are Bloch solutions (see [8]). The spectrum for large \( E \) was also studied; its properties turned out to be similar to those in the case of small coupling [6].

Despite many papers, a complete theory does not exist, and to uncover general properties of almost-periodic operators it is important to study operators with rather general potentials.

In our papers, we began the analysis of the \textit{adiabatic} almost-periodic operator family (1.1). This can be considered as a step in the direction of the investigation of the almost-periodic Schrödinger operators with general potentials. We mostly concentrate on the analysis of the nature of the spectrum. Here, we review the results obtained in [12, 14, 17, 18, 20]. Additional information can be found in [9, 10, 13, 16, 19].

Though, to study (1.1) we use different tools from the theory of almost-periodic operators, our main tools are the \textit{monodromy matrix} and the finite difference \textit{monodromy equation}. These objects are central for the monodromization method, which is a renormalization approach originally developed for studying spectral properties of almost-periodic difference equations (for example, see [1–4]). We extended the monodromization ideas to the case of differential equations (for example, see [12]).

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In studying the almost-periodic operator family, we had to study several different energy regions (intervals of the spectral axis), where the properties of the spectrum are quite different. In each case, using the local (valid on a given compact) asymptotics of the solutions of the almost periodic equation (obtained by means of the asymptotic approach developed in [11, 12, 14, 15, 17, 20]), we described the asymptotic functional structure of the monodromy matrix and, then, studied the monodromy equation, a finite difference equation determined by the monodromy matrix. As a result, it turned out that there are several different model equations, each containing an asymptotic parameter. Each of these equations was efficiently analyzed by means of the methods from the spectral theory of almost-periodic equations.

Our spectral results are described in Secs. 4–8.

In describing the spectral results, we use the terminology and known facts from the theory of the periodic Schrödinger operator. The reader will find them in Sec. 3.

2. A VERY SHORT INTRODUCTION TO THE THEORY OF ALMOST-PERIODIC OPERATORS

Here, we present information necessary for understanding our results. First, following [34], we recall the definitions and the main properties of two central analytic objects of the spectral theory of ergodic Schrödinger operators: the Lyapunov exponent and the integrated density of states. Then, following [12], we describe basic ideas and observations from the monodromization theory.

2.1. Results from the theory of almost-periodic operators

For the sake of definiteness, we discuss only the equation family of the form

\[-\psi''(x) + P(\omega x + \theta)\psi(x) = E\psi(x), \quad x \in \mathbb{R},\] (2.1)

where \(P\) is a real-valued function from \(L^2(\mathbb{T}), \mathbb{T} = [0, 1]^2, \theta \in \mathbb{T}\) is an “ergodic” parameter indexing the equations of the family, and \(\omega = \left(\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right) \in \mathbb{R}^2\) is a fixed “frequency” vector. We assume that \(\omega_1\) and \(\omega_2\) are incommensurate; then (2.1) is an ergodic (or metrically transitive) equation family.

Note that, up to a linear change of variables, (1.1) is of the form (2.1).

Consider the operator \(H(\theta)\) defined by the left-hand side of (2.1). For almost all \(\theta\), it is essentially self-adjoint on \(C_0^\infty(\mathbb{R})\).

**Theorem 2.1.** There exist sets \(\Sigma, \Sigma_{ac}, \text{ and } \Sigma_{sing}\) such that for almost all \(\theta \in \mathbb{R}\), the spectrum, the absolutely continuous spectrum, and the singular spectrum of the operator \(H(\theta)\) coincide with \(\Sigma, \Sigma_{ac}, \text{ and } \Sigma_{sing}\), respectively.

We call \(\Sigma, \Sigma_{ac}, \text{ and } \Sigma_{sing}\), respectively, the spectrum, the absolutely continuous spectrum, and the singular spectrum of the family (2.1).

2.1.1. Integrated density of states

Fix \(L \in \mathbb{N}\). Consider \(H_L(\theta)\) that is the restriction of \(H(\theta)\) to the interval \([-L, L]\) with the Dirichlet boundary conditions at its ends. For almost all \(\theta\), its spectrum is discrete. The limit

\[N(E) = \lim_{L \to \infty} \frac{\text{the number of eigenvalues of } H_L(\theta) \text{ not exceeding } E}{2L + 1}\] (2.2)

exists for almost all \(\theta\) and is independent of \(\theta\). This limit is called the integrated density of states.

**Remark 2.1.** Formula (2.2) defines the integrated density of states for \(P \in L^4_{\text{loc}}\). For \(V \in L^2_{\text{loc}}\), the definition is somewhat different (see the Benderski–Pastur theorem from [34]).

The integrated density of states has the following properties:

- \(N\) is a nondecreasing function of \(E\);
- \(N\) is continuous in \(E\);
- for almost all \(\theta\), the spectrum of \(H(\theta)\) coincides with the closure of the set of points of increase in \(N\).